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SL_n-CHARACTER VARIETIES AS SPACES OF GRAPHS

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ABSTRACT. An SL_n-character of a group G is the trace of an SL_n-representation of G. We show that all algebraic relations between SL_n-characters of G can be visualized as relations between graphs (resembling Feynman diagrams) in any topological space X, with $\pi_1(X) = G$. We also show that all such relations are implied by a single local relation between graphs. In this way, we provide a topological approach to the study of SL_n-representations of groups.

The motivation for this paper was our work with J. Przytycki on invariants of links in 3-manifolds which are based on the Kauffman bracket skein relation. These invariants lead to a notion of a skein module of M which, by a theorem of Bullock, Przytycki, and the author, is a deformation of the SL_2-character variety of $\pi_1(M)$. This paper provides a generalization of this result to all SL_n-character varieties.

1. Introduction

In this paper we introduce a new method in the study of representations of groups into affine algebraic groups. Although we consider only SL_n-representations, the results of this paper can be generalized to other affine algebraic groups; see [Si].

For any group G and any commutative ring R with 1 there is a commutative R-algebra $Rep_n^R(G)$ and *the universal SL_n-representation*

$$j_{G,n} : G \to SL_n(Rep_n^R(G))$$

such that any representation of G into $SL_n(A)$, where A is an R-algebra, factors through $j_{G,n}$ in a unique way. This universal property uniquely determines $Rep_n^R(G)$ and $j_{G,n}$ up to an isomorphism.

$GL_n(R)$ acts on $Rep_n^R(G)$ (see Section 2), and the subring of $Rep_n^R(G)$ composed of the elements fixed by the action, $Rep_n^R(G)^{GL_n(R)}$, is called the *the universal SL_n-character ring of G*. This ring contains essential information about SL_n-representations of G. In particular, if R is an algebraically closed field of characteristic 0, then there are natural bijections between the following three sets:

- the set of all R-algebra homomorphisms $Rep_n^R(G)^{GL_n(R)} \to R$,
- the set of all semisimple $SL_n(R)$-representations of G up to conjugation, and
- the set of $SL_n(R)$-characters of G.

It is convenient to think about $Rep_n^R(G)^{GL_n(R)}$ as the coordinate ring of a scheme, $\mathfrak{X}_n(G) = Spec(Rep_n^R(G)^{GL_n(R)})$, called the SL_n-character variety of G.

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As explained in Section 6, the algebra $Rep_n^R(G)^{GL_n(R)}$ encodes all algebraic relations between the SL_n-characters of G. Unfortunately, it is very difficult to give a finite presentation of $Rep_n^R(G)^{GL_n(R)}$ and, hence, to describe $\mathfrak{X}_n(G)$, even for groups G with relatively simple presentations.

In this paper, we present a topological approach to the study of SL_n-character varieties. We prove that $R[\mathfrak{X}_n(G)] = Rep_n^R(G)^{GL_n(R)}$ is spanned by a special class of graphs (resembling Feynman diagrams) in X, where X is any topological space with $\pi_1(X) = G$; see Theorem 3.7. Moreover, all relations between the elements of this spanning set are induced by specific local relations between the graphs, called *skein relations.*

We postpone a detailed study of applications of our graphical calculus to the theory of SL_n-representations of groups to future papers. In this paper, we content ourself with an example, in which we apply our method to a study of SL_3-representations of the free group on two generators. In this algebraically non-trivial example a huge reduction of computational difficulties can be achieved by the application of our geometric method.

This work is related to several areas of mathematics and physics:

Knot theory (and skein modules). Skein relations between links were used to define the famous polynomial invariants of links, like the Conway, Jones, and Homfly polynomials, [Co, Jo, FYHLMO, P-T, Ka]. In this paper we apply skein relations to the representations of groups.

The motivation for this work was our earlier work on skein modules, [PS-2]. The main theorem of this paper generalizes the Bullock-Przytycki-Sikora theorem relating the Kauffman bracket skein module of a manifold M to the $SL_2(\mathbb{C})$-character variety of $\pi_1(M)$; see [B-2, PS-2].

Quantum invariants of 3-manifolds. We hope that this work will help understand the connections between quantum invariants of 3-manifolds and representations of their fundamental groups. It follows from the work of Yokota [Yo] that for any 3-manifold M, the SU_n-quantum invariants of M can be defined by using our graphs considered up to relations which are q-deformations of our skein relations.

Spin networks and gauge theory. The graphs considered in this paper have an interpretation as spin networks; see [Si]. They are also very similar to graphs used by physicists in non-abelian gauge theory (QCD); see [Cv].

Number theory. After a preliminary version of this paper was made available, M. Kapranov pointed out to us that our work is related to the work of Wiles and others on "pseudo-representations." In his work (related to Fermat's Last Theorem), Wiles gave necessary and sufficient conditions under which a complex-valued function on G is a $GL_2(\mathbb{C})$-character of G. His ideas were developed further and generalized to all GL_n-characters by Taylor, [Ta]. See also [Ny, Ro]. These results provide a description of the coordinate ring of GL_n-character varieties quotiented by nilpotent elements. Our results are similar in spirit, but they are concerned with SL_n-representations and they are stronger, since they describe $R[\mathfrak{X}_n(G)]$ (i.e. $Rep_n^R(G)^{GL_n(R)}$) exactly (with possible nilpotent elements).

The plan of this paper is as follows. In Section 2 we introduce some basic notions and facts concerning representations of groups. In Section 3 we define the algebra $\mathbb{A}_n(X)$ in terms of graphs in X and formulate (Theorems 3.6 and 3.7) the main results of the paper asserting that $\mathbb{A}_n(X)$ is isomorphic to $Rep_n^R(G)^{GL_n(R)}$, where

$G = \pi_1(X)$. The proof requires introducing another algebra, $\mathbb{A}_n(X, x_0)$, associated with any pointed topological space (X, x_0). The algebra $\mathbb{A}_n(X, x_0)$ is an interesting object by itself, and for $n = 2$ it already appeared in the theory of skein modules as a relative skein algebra. Sections 4 and 5 are devoted to the proof of the results of Section 3. In the final section we consider trace identities and use our results to describe the SL_3-character variety of the free group on two generators.

I would like to thank C. Frohman, B. Goldman, and J. H. Przytycki for many helpful conversations. C. Frohman explained some ideas to me which I subsequently used in this paper. Additionally, I would like to thank C. Frohman and W. Goldman for helping me make this paper more "reader-friendly".

2. Background from representation theory

In this section we introduce the basic elements of the theory of SL_n-representations of groups. We follow the approach of Brumfiel and Hilden, ([B-H], Chapter 8), which although formally restricted to SL_2-representations, has a straightforward generalization to SL_n-representations for any n. Compare also [L-M], [Pro-2].

Let G be a group and let R be a commutative ring with unity. There is a commutative R-algebra $Rep_n^R(G)$, called *the universal representation algebra,* and *the universal representation*

$$j_{G,n} : G \to SL_n(Rep_n^R(G)),$$

with the following property: For any commutative R-algebra A and any representation $\rho : G \to SL_n(A)$ there is a unique homomorphism of R-algebras $h_\rho : Rep_n^R(G) \to A$ which induces a homomorphism of groups

$$SL_n(h_\rho) : SL_n(Rep_n^R(G)) \to SL_n(A)$$

such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{j_{G,n}} & SL_n(Rep_n^R(G)) \\ & \overset{\rho}{\searrow} & \big\downarrow {\scriptstyle SL_n(h_\rho)} \\ & & SL_n(A) \end{array}$$

This universal property uniquely determines $Rep_n^R(G)$ up to an isomorphism of R-algebras.

The universal representation algebra of G may also be constructed explicitly in the following way. Let $\langle g_i,\ i \in I | r_j,\ j \in J\rangle$ be a presentation of G such that all relations r_j are monomials in non-negative powers of generators, g_i. Such a presentation exists for every group G. Since we work with groups which are not necessarily finitely presented, I and J may be infinite. Let $P_n(I)$ be the ring of polynomials over R in variables x_{jk}^i, where $i \in I$ and $j, k \in \{1, 2, ..., n\}$. Let A_i, for $i \in I$, be the matrix $(x_{jk}^i) \in M_n(P_n(I))$. For any word $r_j = g_{i_1}^{n_1} g_{i_2}^{n_2} ... g_{i_k}^{n_k}$ consider the corresponding matrix $M_j = A_{i_1}^{n_1} A_{i_2}^{n_2} ... A_{i_k}^{n_k} \in M_n(P_n(I))$. Let $\mathcal{I}$ be the two-sided ideal in $P_n(I)$ generated by polynomials $Det(A_i) - 1$, for $i \in I$, and by all entries of matrices $M_j - Id$, for $j \in J$, where Id is the identity matrix. We denote the quotient $P_n(I)/\mathcal{I}$ by $Rep_n^R(G)$ and the quotient map $P_n(I) \to Rep_n^R(G)$ by η. Let $\bar{x}_{jk}^i = \eta(x_{jk}^i)$ and $\bar{A}_i = (\bar{x}_{jk}^i) \in M_n(Rep_n^R(G))$.

Note that we divided $P_n(I)$ by all relations necessary for the existence of a representation

$$j_{G,n} : G \to SL_n(Rep_n^R(G))$$

such that $j_{G,n}(g_i) = \bar{A}_i$. The algebra $Rep_n^R(G)$ and the representation $j_{G,n} : G \to SL_n(Rep_n^R(G))$ are exactly the universal representation algebra and the universal SL_n-representation of G.

Let $A \in GL_n(R)$. By the definition of the universal representation ring, $Rep_n^R(G)$, there is a unique homomorphism $f_A : Rep_n^R(G) \to Rep_n^R(G)$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{j_{G,n}} & SL_n(Rep_n^R(G)) \\ & \overset{A^{-1}j_{G,n}A}{\searrow} & \Big\downarrow {\scriptstyle SL_n(f_A)} \\ & & SL_n(Rep_n^R(G)) \end{array}$$

One easily observes that f_A is an automorphism of $Rep_n^R(G)$ and that the assignment $A \to f_A$ defines a left action of $GL_n(R)$ on $Rep_n^R(G)$. We denote this action by $A*$, i.e. $f_A(r) = A * r$, for any $r \in Rep_n^R(G)$ and $A \in GL_n(R)$. We call the ring $Rep_n^R(G)^{GL_n(R)}$ consisting of elements of $Rep_n^R(G)$ fixed by the action of $GL_n(R)$ *the universal character ring of* G. This term indicates a connection between the ring $Rep_n^R(G)^{GL_n(R)}$ and SL_n-characters of G, i.e. traces of SL_n-representations of G. In the simplest case, when R is an algebraically closed field of characteristic 0, and G is finitely generated, this connection can be described as follows. Every representation $\rho : G \to SL_n(R)$ induces a homomorphism $h_\rho : Rep_n^R(G) \to R$, whose restriction to $Rep_n^R(G)^{GL_n(R)}$ we denote by h'_ρ. The next proposition follows from geometric invariant theory and from the results of [L-M].

Proposition 2.1. *Under the above assumptions, the following sets are in a natural correspondence given by bijections* $\rho \to h'_\rho$, *and* $\rho \to \chi = tr \circ \rho$:

- *the set of all semisimple* SL_n*-representations of* G;
- *the set of all* R*-homomorphisms* $Rep_n^R(G)^{GL_n(R)} \to R$;
- *the set of all* $SL_n(R)$*-characters of* G.

By the proposition, we can identify the above sets and denote them by $X_n(G)$. By the second definition, $X_n(G)$ is the affine algebraic set composed of the closed points of the SL_n-character variety, $\mathfrak{X}_n(G)$, defined in the introduction. In other words,

$$R[X_n(G)] = R[\mathfrak{X}_n(G)]/\sqrt{0} = Rep_n^R(G)^{GL_n(R)}/\sqrt{0}.$$

As shown in [L-M, KM], $Rep_n^R(G)^{GL_n(R)}$ may contain nilpotent elements and, therefore, $\mathfrak{X}_n(G)$ contains more subtle information about SL_n-representations of G than $X_n(G)$. The ring $Rep_n^R(G)^{GL_n(R)}$ will be given a topological description in Section 3.

The $GL_n(R)$ action on $Rep_n^R(G)$ induces an action of $GL_n(R)$ on the ring of $n \times n$ matrices over $Rep_n^R(G)$. If $M = (m_{ij}) \in M_n(Rep_n^R(G))$ and $A \in GL_n(R)$, then

$$A * M = A \begin{pmatrix} A*m_{11} & A*m_{12} & ... & A*m_{1n} \\ \vdots & \vdots & ... & \vdots \\ A*m_{n1} & A*m_{n2} & ... & A*m_{nn} \end{pmatrix} A^{-1}. \tag{2.1}$$

There is an equivalent definition of the action of $GL_n(R)$ on $Rep_n^R(G)$ and on $M_n(Rep_n^R(G))$. In order to introduce it we will first define $GL_n(R)$-actions on $P_n(I)$ and $M_n(P_n(I))$. We can consider $P_n(I)$ as a ring of polynomial functions defined on the product of I copies of $M_n(R)$, $M_n(R)^I \to R$, by identifying $x_{jk}^{i_0} \in P_n(I)$ with a map assigning to $(M_i)_{i\in I} \in M_n(R)^I$ the (j,k)-entry of M_{i_0}. Therefore,

$$P_n(I) = Poly(M_n(R)^I, R).$$

With this identification any entry in a matrix M in $M_n(P_n(I))$ is a polynomial function on $M_n(R)^I$. Therefore we can think of elements of $M_n(P_n(I))$ as coordinatewise polynomial functions $M_n(R)^I \to M_n(R)$,

$$M_n(P_n(I)) = Poly(M_n(R)^I, M_n(R)).$$

If X, Y are sets with a left G-action, then the set of all functions $Fun(X,Y)$ has a natural left G-action defined for any $f : X \to Y$ and $g \in G$ by $g * f(x) = gf(g^{-1}x)$, for $x \in X$. GL_n acts on $M_n(R)$ and on $M_n(R)^I$ by conjugation, and it acts trivially on R. These actions induce $GL_n(R)$-actions on $Fun(M_n(R)^I, R)$ and on $Fun(M_n(R)^I, M_n(R))$, which restrict to

$$P_n(I) = Poly(M_n(R)^I, R)$$

and

$$M_n(P_n(I)) = Poly(M_n(R)^I, M_n(R)).$$

The following statement is a consequence of the above definitions.

Lemma 2.2. (1) *The natural embedding of $P_n(I)$ into $M_n(P_n(I))$ as scalar matrices is $GL_n(R)$-equivariant.*
(2) *$A_{i_0} = (x_{jk}^{i_0}) \in M_n(P_n(I))$ is invariant under the action of $GL_n(R)$, for any $i_0 \in I$.*

Now, we are going to show that $\eta : P_n(I) \to Rep_n^R(G)$ is $GL_n(R)$-equivariant, and hence the action of $GL_n(R)$ on $P_n(I)$ induces a $GL_n(R)$-action on $Rep_n^R(G)$ which coincides with the $GL_n(R)$-action on $Rep_n^R(G)$ defined previously.

Proposition 2.3. *The following diagram commutes:*

$$\begin{array}{ccc} M_n(P_n(I)) & \xrightarrow{M_n(\eta)} & M_n(Rep_n^R(G)) \\ \downarrow Tr & & \downarrow Tr \\ P_n(I) & \xrightarrow{\eta} & Rep_n^R(G) \end{array} \qquad (2.2)$$

and all maps appearing in it intertwine with the $GL_n(R)$-action.

Proof. Since the commutativity of the above diagram is obvious, we will prove only that the trace functions and homomorphisms η, $M_n(\eta)$ are $GL_n(R)$-equivariant.

- The trace map $Tr : M_n(R) \to R$ is $GL_n(R)$-equivariant. Therefore the induced map

$$Tr : M_n(P_n(I)) = Poly(M_n(R)^I, M_n(R)) \to Poly(M_n(R)^I, R) = P_n(I)$$

is also $GL_n(R)$-equivariant.
- If $M = (m_{ij}) \in M_n(Rep_n^R(G))$ and $A \in GL_n(R)$, then $A * M$ is given by matrix (2.1), whose trace is $Tr(A*M) = \sum_{i=1}^n A*m_{ii} = A*Tr(M)$. Therefore

$$Tr : M_n(Rep_n^R(G)) \to Rep_n^R(G)$$

is $GL_n(R)$-equivariant.

- Recall that $P_n(I)$ is generated by elements x^i_{jk}. Therefore in order to prove that η is $GL_n(R)$-equivariant it is enough to show that $\eta(A * x^i_{jk}) = A * \bar{x}^i_{jk}$, for any $i \in I,\ j,k \in \{1,2,...,n\}$, where $\bar{x}^i_{jk} = \eta(x^i_{jk}) \in Rep^R_n G$.

  For any $i_0 \in I,\ j_{G,n}(g_{i_0}) = \bar{A}_{i_0} \in SL_n(Rep^R_n(G))$. By the definition of the $GL_n(R)$-action on $Rep^R_n(G)$, $A^{-1} j_{G,n}(g_{i_0}) A$ is the matrix obtained from $j_{G,n}(g_{i_0})$ by acting on all its entries by A. Therefore

$$A * \bar{x}^{i_0}_{jk} = (j,k)\text{-entry of } A^{-1}\bar{A}_{i_0}A. \tag{2.3}$$

  Having described $A*\bar{x}^{i_0}_{jk}$, we need to give an explicit description of $A*x^{i_0}_{jk} \in P_n(I)$. Recall that we identified $x^{i_0}_{jk}$ with the map $M_n(R)^I \to R$ assigning to $\{M_i\}_{i\in I}$ the (j,k)-entry of M_{i_0}. The definition of the $GL_n(R)$-action on maps between $GL_n(R)$-sets implies that

$$(A * x^{i_0}_{jk})(\{M_i\}_{i\in I}) = A * \left(x^{i_0}_{jk}(A^{-1} * \{M_i\}_{i\in I})\right).$$

  Since $GL_n(R)$ acts by simultaneous conjugation on $M_n(R)^I$ and it acts trivially on R, the right side of the above equation is equal to the (j,k)-entry of $A^{-1}M_{i_0}A$. But the entries of M_{i_0} are given by the values of functions $x^{i_0}_{jk}$ evaluated on $\{M_i\}_{i\in I}$. Therefore

$$A * x^{i_0}_{jk} = (j,k)\text{-entry of } A^{-1}(x^{i_0}_{jk})A. \tag{2.4}$$

  Finally, (2.3) and (2.4) imply that

$$\eta(A * x^{i_0}_{jk}) = \eta\left(\text{the } (j,k)\text{-entry of } A^{-1}A_{i_0}A\right)$$
$$= \text{the } (j,k)\text{-entry of } A^{-1}\bar{A}_{i_0}A = A * \bar{x}^{i_0}_{jk}.$$

- We prove that $M_n(\eta)$ is equivariant. Let $M = (m_{jk}) \in M_n(P_n(I))$. Notice that the definition of $GL_n(R)$-action on $M_n(P_n(I))$ implies that $A * M = A\,(A * m_{jk})\,A^{-1}$. Therefore

$$M_n(\eta)(A * M) = M_n(\eta)(A(A * m_{jk})A^{-1}) = A(\eta(A * m_{jk}))A^{-1}.$$

  Since η is $GL_n(R)$-equivariant, the matrix on the right side of the above equation is $A(A * \eta(m_{jk}))A^{-1} = A * (\eta(m_{jk}))$. Therefore $M_n(\eta)$ is also $GL_n(R)$-equivariant. □

The above proposition implies that there exists a function

$$Tr : M_n(Rep^R_n(G))^{GL_n(R)} \to Rep^R_n(G)^{GL_n(R)}.$$

This function will be given a simple topological interpretation in the next section.

Proposition 2.4. *The image of the universal representation*

$$j_{G,n} : G \to M_n(Rep^R_n(G))$$

is invariant under the action of $GL_n(R)$.

Proof. Since the elements g_i generate G, it is sufficient to show that $j_{G,n}(g_i) \in M_n(Rep^R_n(G))^{GL_n(R)}$. By Lemma 2.2(2), $A_i \in M_n(P_n(I))^{GL_n(R)}$. The map $M_n(\eta)$ is equivariant. Therefore it takes the invariant A_i to the invariant $M_n(\eta)(A_i) = j_{G,n}(g_i)$. □

3. Skein algebras

In this section we assign to each path-connected topological space X a commutative R-algebra $\mathbb{A}_n(X)$ and to each pointed path-connected topological space (X, x_0) an R-algebra $\mathbb{A}_n(X, x_0)$. These algebras encode the most important information about the SL_n-representations of $\pi_1(X, x_0)$. We will show that if R is a field of characteristic 0 (but not necessarily algebraically closed), then $\mathbb{A}_n(X)$ is isomorphic to the universal character ring $Rep_n^R(G)^{GL_n(R)}$, where $G = \pi_1(X, x_0)$, and $\mathbb{A}_n(X, x_0)$ is isomorphic to $M_n(Rep_n^R(G))^{GL_n(R)}$.

We start with a definition of a graph which is the most suitable for our purposes. A *graph* $D = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ consists of a vertex-set $\mathcal{V}$, a set of oriented edges $\mathcal{E}$, and a set of oriented loops $\mathcal{L}$. Each edge $E \in \mathcal{E}$ has a beginning $b(E) \in \mathcal{V}$ and an end $e(E) \in \mathcal{V}$. Loops have neither beginnings nor ends. If $b(E) = v$ or $e(E) = v$, then E is *incident* with v. For any $v \in \mathcal{V}$, all edges incident to v are ordered by consecutive integers $1, 2, \ldots$. Therefore the beginning and the end of each edge is assigned a number.

The sets $\mathcal{V}, \mathcal{E}, \mathcal{L}$ are finite. We topologize each graph as a CW-complex. The topology of a graph coincides with the topology of its edges $E \simeq [0,1], E \in \mathcal{E}$, and its loops $L \simeq S^1, L \in \mathcal{L}$. There is a natural notion of isomorphism of graphs.

Let $\mathcal{G}$ be a set of representatives of all isomorphism classes of graphs defined above. We say that a vertex v is an n-valent source of a graph D if n distinct edges of D begin at v and no edge ends at v. Similarly, we say that v is an n-valent sink of D if n distinct edges end at v and no edge begins at v. Let $\mathcal{G}_n$ denote the set of all graphs in $\mathcal{G}$, all of whose vertices are either n-valent sources or n-valent sinks. We assume that the empty graph $\emptyset$ is also an element of $\mathcal{G}_n$. We denote the single loop in $\mathcal{G}_n$, i.e. the connected graph without any vertices, by S^1. Let $\mathcal{G}_n'$ denote the set of all graphs $D \in \mathcal{G}$ such that D has one 1-valent source and one 1-valent sink, and all other vertices of D are n-valent sources or n-valent sinks. We denote the single edge in $\mathcal{G}_n'$, i.e. the connected graph without any n-valent vertices, by $[0,1]$.

Let X be a path-connected topological space. We will call any continuous map $f : D \to X$, where $D \in \mathcal{G}_n$, *a graph in* X. We identify two maps $f_1, f_2 : D \to X$ if they are homotopic. Let us denote the set of all graphs in X by $\mathcal{G}_n(X)$. Similarly, we define $\mathcal{G}_n(X, x_0)$ to be the set of all maps $f : D \to X \times [0,1]$, where $D \in \mathcal{G}_n'$ and f maps the 1-valent sink of D to $(x_0, 1)$ and the 1-valent source of D to $(x_0, 0)$. We identify maps which are homotopic relative to $(x_0, 0)$ and $(x_0, 1)$. We will call elements of $\mathcal{G}_n(X, x_0)$ *relative graphs in* $X \times [0,1]$.

We introduce a few classes of graphs in $\mathcal{G}_n(X)$ and $\mathcal{G}_n(X, x_0)$ which will often be used later on in the paper. Let $L_\gamma : S^1 \to X$ be a graph in X which represents the conjugacy class of $\gamma \in \pi_1(X, x_0)$. We denote by E_γ a relative graph $E_\gamma : [0,1] \to X \times [0,1],\ E_\gamma(0) = (x_0, 0),\ E_\gamma(1) = (x_0, 1)$, whose projection into X,

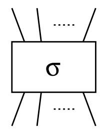
$$[0,1] \stackrel{E_\gamma}{\to} X \times [0,1] \to X,$$

represents $\gamma \in \pi_1(X, x_0)$. Let $EL_\gamma : [0,1] \cup S^1 \to X \times [0,1]$ be a relative graph such that $EL_\gamma(t) = (x_0, t)$ for $t \in [0,1]$, and $EL_{\gamma|S^1} : S^1 \to X \times [0,1] \to X$ represents the conjugacy class of $\gamma \in \pi_1(X, x_0)$.

For any two graphs $f_1 : D_1 \to X$ and $f_2 : D_2 \to X,\ f_1, f_2 \in \mathcal{G}_n(X)$, we define a product of them to be $f_1 \cup f_2 : D_1 \cup D_2 \to X,\ f_1 \cup f_2 \in \mathcal{G}_n(X)$, where $D_1 \cup D_2$ denotes the disjoint union of D_1 and D_2. Therefore the free R-module $R\mathcal{G}_n(X)$ on

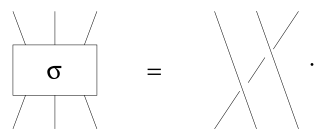
$\mathcal{G}_n(X)$ can be considered as a commutative R-algebra. The empty graph $\emptyset : \emptyset \to X$ is an identity in $R\mathcal{G}_n(X)$.

In the next definition we will represent fragments of diagrams by coupons, as depicted below:

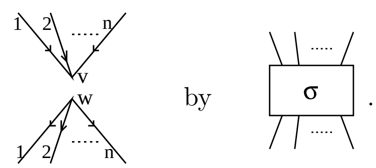


This coupon means a braid corresponding to a permutation $\sigma \in S_n$.[1]

Example 3.1. If $\sigma = (1, 2, 3) \in S_3$, then



Suppose that $f : D \to X$ is a graph in X and f maps a source w and a sink v of D to the same point $x_1 \in X$. Let D_σ be a graph obtained from D by replacing



In the diagram we display v and w as separate points to accentuate the fact that they are distinct in the domain of the mapping f. There is an obvious way to modify $f : D \to X$ to a map $f_\sigma : D_\sigma \to X$. We call $(f, \{f_\sigma\}_{\sigma \in S_n})$ a family of *skein related graphs at* x_1.

Definition 3.2. Let X be a path-connected topological space, and let I be the ideal in $R\mathcal{G}_n(X)$ generated by two kinds of expressions:

(1) $f - \sum_{\sigma \in S_n} \epsilon(\sigma) f_\sigma$, where $\epsilon(\sigma)$ denotes the sign of σ and $(f, \{f_\sigma\}_{\sigma \in S_n})$ is a family of skein related graphs at some point $x_1 \in X$.

(2) $L_e - n$, where e is the identity element in $\pi_1(X, x_0)$ (i.e. L_e is a homotopically trivial loop).

Then the R-algebra $\mathbb{A}_n(X) = R\mathcal{G}_n(X)/I$ is called the n-th skein algebra of X.

Similarly we define $\mathbb{A}_n(X, x_0)$. Let $f_1 : D_1 \to X \times [0, 1], f_2 : D_2 \to X \times [0, 1]$ be elements of $\mathcal{G}_n(X, x_0)$. We define the product of them to be a map $f_1 \cdot f_2 : D_1 \cup D_2 \to X \times [0, 1]$, such that

$$(f_1 \cdot f_2)(d) = \begin{cases} (x, \frac{1}{2}t) & \text{if } d \in D_1 \text{ and } f_1(d) = (x, t), \\ (x, \frac{1}{2}t + \frac{1}{2}) & \text{if } d \in D_2 \text{ and } f_2(d) = (x, t). \end{cases}$$

This product extends to an associative (but generally non-commutative) product in $R\mathcal{G}_n(X, x_0)$. The identity in $R\mathcal{G}_n(X, x_0)$ is a map $f : E \to X \times [0, 1]$, where E is a single edge and f maps E onto $\{x_0\} \times [0, 1]$.

[1] Since we consider graphs up to homotopy equivalence, it does not matter which braid corresponding to σ we take.

If $f : D \to X \times [0,1]$ is an element of $\mathcal{G}_n(X, x_0)$ such that an n-valent source w and an n-valent sink v of D are mapped to a point $x_1 \in X \times [0,1]$, then one can define D_σ and $f_\sigma : D_\sigma \to X \times [0,1]$, $f_\sigma \in \mathcal{G}_n(X, x_0)$, in exactly the same way as it was done for graphs in $\mathcal{G}_n(X)$ in the paragraph preceding Definition 3.2. We say, as before, that $(f, \{f_\sigma\}_{\sigma \in S_n})$ are graphs skein related at x_1.

Definition 3.3. Let X be a path-connected topological space with a specified point $x_0 \in X$, and let I' be the ideal in $R\mathcal{G}_n(X, x_0)$ generated by expressions

(1) $f - \sum_{\sigma \in S_n} \epsilon(\sigma) f_\sigma$, where $(f, \{f_\sigma\}_{\sigma \in S_n})$ is a family of skein related graphs at some point $x_1 \in X \times [0,1]$.
(2) $EL_e - n$, where e is the identity in $\pi_1(X, x_0)$.

Then the R-algebra $\mathbb{A}_n(X, x_0) = R\mathcal{G}_n(X, x_0)/I'$ is called the *n-th relative skein algebra* of (X, x_0).

Note that different choices of $x_0 \in X$ give isomorphic algebras $\mathbb{A}_n(X, x_0)$.

Let $f \in \mathcal{G}_n(X)$, $f : D \to X$. Let $D' = D \cup E$, where E is an edge disjoint from D. Then D' has one 1-valent sink e_0 and one 1-valent source e_1, $\{e_0, e_1\} = \partial E$, and $D' \in \mathcal{G}'_n$. We extend f to $f' : D' \to X \times [0,1]$ in such a way that $f'(d) = (f(d), \frac{1}{2})$ for $d \in D$, and $f'(t) = (x_0, t)$ for $t \in [0,1] \simeq E$, where $[0,1] \simeq E$ is an orientation-preserving parameterization of E. This operation defines an embedding $\imath : \mathcal{G}_n(X) \to \mathcal{G}_n(X, x_0)$, $\imath(f) = f'$. Notice that $\imath$ induces a homomorphism $\imath_* : \mathbb{A}_n(X) \to \mathbb{A}_n(X, x_0)$. Therefore we can consider $\mathbb{A}_n(X, x_0)$ as an $\mathbb{A}_n(X)$-algebra.

Let $f : D \to X \times [0,1]$ be a map, $f \in \mathcal{G}_n(X, x_0)$. Let $\overline{D} \in \mathcal{G}_n$ be a graph obtained by identification of the 1-valent sink with the 1-valent source in D. Let us compose $f : D \to X \times [0,1]$ with a projection $X \times [0,1] \to X$. This composition gives a map $\overline{f} : \overline{D} \to X$, $\overline{f} \in \mathcal{G}_n(X)$. Therefore, we have a function $\bar{\cdot} : \mathcal{G}_n(X, x_0) \to \mathcal{G}_n(X)$. This function can be extended to an R-linear homomorphism $\mathbb{T} : \mathbb{A}_n(X, x_0) \to \mathbb{A}_n(X)$. Notice that for any graph $D \in \mathcal{G}_n(X)$, $\mathbb{T}(\imath_*(D))$ is equal to a union of $f : D \to X$ with a contractible loop in X. Hence by Definition 3.2(2) $\mathbb{T}(\imath_*(D)) = n \cdot D$. Since graphs in X span $\mathbb{A}_n(X)$, the composition of $\imath_* : \mathbb{A}_n(X) \to \mathbb{A}_n(X, x_0)$ with $\mathbb{T} : \mathbb{A}_n(X, x_0) \to \mathbb{A}_n(X)$ is equal to n times the identity on $\mathbb{A}_n(X)$. This implies the following fact.

Fact 3.4. *If $\frac{1}{n} \in R$, then $\imath_* : \mathbb{A}_n(X) \to \mathbb{A}_n(X, x_0)$ is a monomorphism of rings.*

The next proposition summarizes basic properties of $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$.

Proposition 3.5. (1) *The assignment $X \to \mathbb{A}_n(X)$ (respectively, $(X, x_0) \to \mathbb{A}_n(X, x_0)$) defines a functor from the category of path-connected topological spaces (respectively, category of path-connected pointed spaces) to the category of commutative R-algebras (respectively, the category of R-algebras).*
(2) *If $f : X \to Y, f(x_0) = y_0$, induces a surjection $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, then the corresponding homomorphisms $\mathbb{A}_n(f) : \mathbb{A}_n(X, x_0) \to \mathbb{A}_n(Y, y_0)$ and $\mathbb{A}_n(f) : \mathbb{A}_n(X) \to \mathbb{A}_n(Y)$ are epimorphisms of R-algebras.*
(3) *The algebra $\mathbb{A}_n(X)$ is generated by loops in X, i.e. by graphs L_γ, for $\gamma \in \pi_1(X, x_0)$.*
(4) *The algebra $\mathbb{A}_n(X, x_0)$ is generated by graphs $E_{g_i^{\pm 1}}$ and EL_γ, where $\{g_i\}_{i \in I}$ is a set of generators of $\pi_1(X, x_0)$ and $\gamma \in \pi_1(X, x_0)$.*

Proof. Since statements (1) and (2) of Proposition 3.5 are obvious, we give a proof of (3) and (4) only.

From the definition of a graph $D \in \mathcal{G}_n$ or $D \in \mathcal{G}'_n$ we see that it has an equal number of n-valent sinks and sources. Relation (1) of Definition 3.2 and of Definition 3.3 implies that each pair of vertices of $f : D \to X,\ f \in \mathcal{G}_n(X)$ (respectively, of $f : D \to X \times [0,1],\ f \in \mathcal{G}_n(X, x_0)$) composed of a sink and a source can be resolved and f can be replaced by a linear combination of graphs with a smaller number of sinks and sources. Therefore, after a finite number of steps each graph in $\mathcal{G}_n(X)$ (respectively, $\mathcal{G}_n(X, x_0)$) can be expressed as a linear combination of graphs without n-valent vertices.

(3) If $f : D \to X,\ f \in \mathcal{G}_n(X)$, and D has no vertices, then D is a union of loops, $D = S^1 \cup S^1 \cup ... \cup S^1$, and therefore $f = L_{\gamma_1} \cdot L_{\gamma_2} \cdot ... \cdot L_{\gamma_k} \in \mathbb{A}_n(X)$, for some $\gamma_1, \gamma_2, ..., \gamma_k \in \pi_1(X, x_0)$.

(4) If $f : D \to X \times [0,1],\ f \in \mathcal{G}_n(X, x_0)$, and D has no n-valent vertices, then $D = [0,1] \cup S^1 \cup S^1 \cup ... \cup S^1$. Suppose that $[0,1] \xrightarrow{f} X \times [0,1] \to X$ represents $\gamma_0 \in \pi_1(X, x_0)$, and f restricted to the j-th circle represents the conjugacy class of a $\gamma_j \in \pi_1(X, x_0),\ j = 1, 2, ..., k$. Then $f = E_{\gamma_0} \cdot EL_{\gamma_1} \cdot EL_{\gamma_2} \cdot ... \cdot EL_{\gamma_k} \in \mathbb{A}_n(X, x_0)$. Therefore $\mathbb{A}_n(X, x_0)$ is generated by the elements EL_γ and $E_{\gamma'},\ \gamma, \gamma' \in \pi_1(X, x_0)$. But each $E_{\gamma'}$ is a product of elements $E_{g_i^{\pm 1}}$, where $\{g_i\}$ is a set of generators of $\pi_1(X, x_0)$. □

We will see later that $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$ depend only on $\pi_1(X, x_0)$. Moreover, if $\pi_1(X, x_0)$ is a finitely generated group, then the algebras $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$ are also finitely generated.

Now we are ready to formulate the most important results of this paper.

Theorem 3.6. *Let X be any (path-connected) topological space, and let $G = \pi_1(X, x_0), x_0 \in X$. There are R-algebra homomorphisms*

$$\Theta : \mathbb{A}_n(X, x_0) \to M_n(Rep_n^R(G))^{GL_n(R)}, \qquad \theta : \mathbb{A}_n(X) \to Rep_n^R(G)^{GL_n(R)},$$

uniquely determined by the following conditions:

(1) $\Theta(E_\gamma) = j_{G,n}(\gamma)$ *and* $\Theta(EL_{\gamma'}) = Tr(j_{G,n}(\gamma'))$, *for any* $\gamma, \gamma' \in \pi_1(X, x_0)$.
(2) $\theta(L_\gamma) = Tr(j_{G,n}(\gamma))$, *for any* $\gamma \in \pi_1(X, x_0)$.

Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{A}_n(X, x_0) & \xrightarrow{\Theta} & M_n(Rep_n^R(G))^{GL_n(R)} \\ \downarrow \mathbb{T} & & \downarrow Tr \\ \mathbb{A}_n(X) & \xrightarrow{\theta} & Rep_n^R(G)^{GL_n(R)} \end{array} \qquad (3.1)$$

Theorem 3.7. *If R is a field of characteristic 0, then*

$$\Theta : \mathbb{A}_n(X, x_0) \to M_n(Rep_n^R(G))^{GL_n(R)}, \qquad \theta : \mathbb{A}_n(X) \to Rep_n^R(G)^{GL_n(R)}$$

are isomorphisms of R-algebras.

Let R be a field of characteristic 0. It can be shown that if X is a 3-manifold, then $\mathbb{A}_2(X)$ is isomorphic to the Kauffman bracket skein module of X, $\mathcal{S}_{2,\infty}(X, R, \pm 1)$. Moreover, if X is a surface, then $\mathbb{A}_2(X, x_0)$ is isomorphic to the relative Kauffman bracket skein module of X, $\mathcal{S}^{rel}_{2,\infty}(X, R, \pm 1)$. See [PS-2], [H-P], for appropriate definitions and the notational conventions. The main results of [B-1], [B-2], [PS-1] and [PS-2] relate the Kauffman bracket skein modules of 3-manifolds with the SL_2-representation theory of their fundamental groups. Theorem 3.7 generalizes these results to groups SL_n, for any n.

Moreover, it can be shown that in the case when X is any path-connected topological space, $\mathbb{A}_2(X, x_0)$ and $\mathbb{A}_2(X)$ can be given the following simple algebraic description: Let $G = \pi_1(X)$ and let I be the ideal in the group ring RG generated by elements $h(g + g^{-1}) - (g + g^{-1})h$, where $g, h \in G$. There is an involution τ on $H(G) = RG/I$ sending g to g^{-1}. One can show that $\mathbb{A}_2(X, x_0)$ is isomorphic to $H(G)$ and $\mathbb{A}_2(X)$ is isomorphic to $H^+(G)$, where $H^+(G)$ is the subring of $H(G)$ invariant under τ. The algebras $H(G), H^+(G)$ are introduced and thoroughly investigated in [B-H]. One of the main results of [B-H] is that $H(G) = M_n(Rep_n^R(G))^{GL_n(R)}$ and $H^+(G) = Rep_n^R(G)^{GL_n(R)}$, for $n = 2$. (Compare also [Sa-1], [Sa-2].) Theorem 3.7 can be considered as a generalization of this result to all values of n.

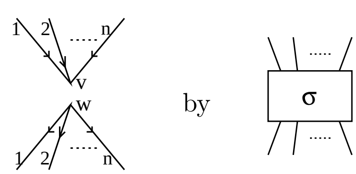
4. Proof of Theorem 3.6

Before we prove Theorem 3.6 we give new definitions of $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$ which only use $G = \pi_1(X, x_0)$.

Let X be a path-connected topological space and $x_0 \in X$. For any graph in $\mathcal{G}_n(X)$, i.e. a map $f : D \to X$ for some $D \in \mathcal{G}_n$, there is a map $f' : D \to X$ homotopic to f, which maps all vertices of D to x_0. Therefore the homotopy class of f can be described by the graph D with each edge E labeled by an element of $\pi_1(X, x_0)$ corresponding to the map $f'_{|E} : E \to X$ and each loop L labeled by the conjugacy class in $\pi_1(X, x_0)$ corresponding to the map $f'_{|L} : L \simeq S^1 \xrightarrow{f} X$. This description does not need to be unique.

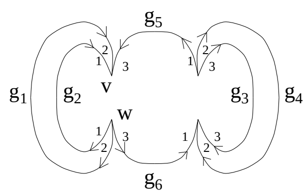
We denote the set of graphs in $\mathcal{G}_n$ all of whose edges are labeled by elements of G and all loops are labeled by conjugacy classes in G by $\mathcal{G}_n(G)$. There is a natural multiplication operation on $\mathcal{G}_n(G)$. The product of $D_1, D_2 \in \mathcal{G}_n(G)$ is the disjoint union of D_1 and D_2. Therefore $R\mathcal{G}_n(G)$ is a commutative R-algebra with $\emptyset$ as the identity.

Let D be a graph in $\mathcal{G}_n(G)$. We have noticed already that D corresponds to a map $f : D \to X$ which maps all vertices of D to x_0 and restricted to edges and loops of D agrees with their labeling. Such f is unique up to a homotopy which fixes the vertices of D. Let w be a source and v be a sink in D. Since f maps v and w to the same point in X, there exists a map $f_\sigma : D_\sigma \to X$ defined for any $\sigma \in S_n$ as in the paragraph preceding Definition 3.2. Notice that f_σ maps all vertices of D_σ to $x_0 \in X$. Therefore, we can label all edges of D_σ by appropriate elements of G and all loops of D_σ by appropriate conjugacy classes in G, and hence consider D_σ as an element of $\mathcal{G}_n(G)$. Hence, we have showed that one can replace any source w and any sink v in an arbitrary graph $D \in \mathcal{G}_n(G)$

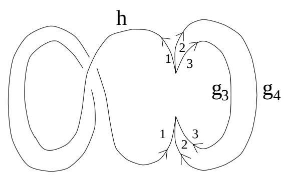


and obtain a well-defined graph $D_\sigma \in \mathcal{G}_n(G)$.

As an example consider the graph D presented below:



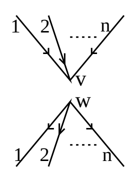
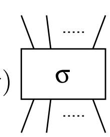
Replacing the vertices v, w by a coupon decorated by $\sigma = (123) \in S_3$ gives a diagram D_σ:

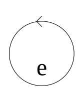


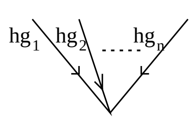
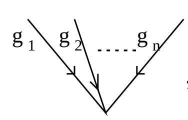
where $h = g_6 g_1 g_2 g_5$.

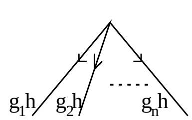
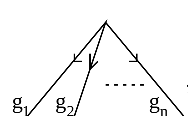
Now we are ready to define $\mathbb{A}_n(X)$ in terms of graphs in $\mathcal{G}_n(G)$. Namely, this algebra is isomorphic to $R\mathcal{G}_n(\pi_1(X, x_0))/I$, where $I \triangleleft R\mathcal{G}_n(\pi_1(X, x_0))$ is an ideal generated by relations analogous to relations (1) and (2) of Definition 3.2 and by relations following from the fact that the assignment $\mathcal{G}_n(G) \to \mathcal{G}_n(X)$ described above is onto but not 1-1. The problem comes from the fact that one can take a graph $D \in \mathcal{G}_n$ whose edges and loops are labeled in two different ways such that the corresponding maps $f, f' : D \to X$ sending the vertices of D to x_0 are homotopic but not by a homotopy relative to the vertices of D. In order to resolve this problem we need to allow an operation which moves vertices of D around paths in X beginning and ending at x_0. Notice however that it suffices to move one vertex at a time. The following fact summarizes our observations.

Fact 4.1. *Let X be a path-connected topological space with a specified point $x_0 \in X$, and let $G = \pi_1(X, x_0)$. Let I be the ideal in $R\mathcal{G}_n(G)$ generated by expressions of the following form:*

(4.1)  $- \sum_{\sigma \in S_n} \epsilon(\sigma)$  ,

(4.2)  $- n,$

(4.3)  $-$  ,

(4.4)  $-$  ,

for any $g_1, g_2, ..., g_n, h \in G$.

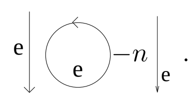
Then there is an isomorphism between the R-algebras $\mathbb{A}_n(X)$ and $R\mathcal{G}_n(G)/I$ assigning to each graph $f : D \to X, f \in \mathcal{G}_n(X)$, with all vertices at x_0 the graph D with every edge E of D decorated by the element of $\pi_1(X, x_0)$ corresponding to $f_{|E} : E \to X$, and every loop L of D decorated by the conjugacy class in $\pi_1(X, x_0)$ represented by $f_{|L} : L \to X$.

Now we will state a similar fact for $\mathbb{A}_n(X, x_0)$. Let $\mathcal{G}'_n(G)$ be a set of graphs in $\mathcal{G}'_n$ all of whose edges are labeled by elements of G and all of whose loops are labeled by conjugacy classes in G.

There is a multiplication operation defined on $\mathcal{G}'_n(G)$ in the following way. Let $D_1, D_2 \in \mathcal{G}'_n(G)$, let v_i be the 1-valent source of D_i, $i \in \{1, 2\}$, and let w_i be the 1-valent sink of D_i. Let g_i be the label of the edge of D_i joining v_i with w_i. The graph $D_1 \cdot D_2$ is obtained from the disjoint union of D_1 and D_2 by identifying v_1 with w_2. The edge of $D_1 \cdot D_2$ joining v_2 with w_1 is labeled by $g_1 \cdot g_2$. All other edges and loops of $D_1 \cdot D_2$ inherit labels from D_1 and D_2. A single edge labeled by $e \in G$ is the identity in $\mathcal{G}'_n(G)$.

This multiplication extends to an associative (but not commutative) multiplication in $R\mathcal{G}'_n(G)$.

Fact 4.2. *Let X be a path-connected topological space with a specified point $x_0 \in X$, and let $G = \pi_1(X, x_0)$. Let I' be the ideal in $R\mathcal{G}'_n(G)$ generated by the expressions* (4.1),(4.3), (4.4) *and*

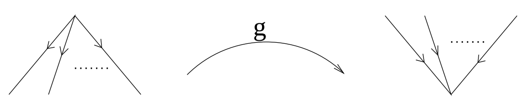


Then $\mathbb{A}_n(X, x_0) \simeq R\mathcal{G}'_n(G)/I'$.

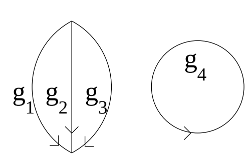
Facts 4.1 and 4.2 show that the algebras $\mathbb{A}_n(X, x_0)$ and $\mathbb{A}_n(X)$ depend only on $\pi_1(X, x_0)$. In fact 4.1 and 4.2 give us models for $\mathbb{A}_n(X, x_0)$ and $\mathbb{A}_n(X)$ built from $\mathcal{G}_n(G)$ and $\mathcal{G}'_n(G)$. In the rest of this section we will use these models.

Let us fix a commutative ring R and a positive integer n and a topological space X with $x_0 \in X, \pi_1(X, x_0) = G$. Let $\mathcal{R} = Rep_n^R(G)$ and let $V = \mathcal{R}^n$ be a free n-dimensional module over $\mathcal{R}$ with the standard basis, $\{e_1, e_2, ..., e_n\}$, $e_i = (0, 0, ..., 1, ..., 0)$. The dual space V^* has the dual basis $e^1, e^2, ..., e^n$, $e^i(e_j) = \delta_{i,j}$. We will always use the standard bases and therefore identify $V^* \otimes V \simeq End_{\mathcal{R}}(V) \simeq M_n(\mathcal{R})$.

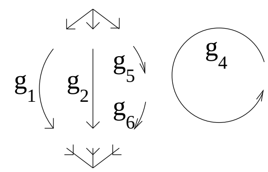
Let D be an element of $\mathcal{G}_n(G)$ or $\mathcal{G}'_n(G)$. We can decompose D into arcs, sources and sinks:



Example 4.3.



can be decomposed to



where $g_1, g_2, g_3, g_4, g_5, g_6 \in G,\ g_3 = g_6 g_5$.

Notice that the decomposition of a graph is not unique, since we can cut each edge or loop into many pieces.

Let us assign to each n-valent source the tensor

$$\sum_{\sigma \in S_n} \epsilon(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes ... \otimes e_{\sigma(n)} \in V^{\otimes n},$$

and to each n-valent sink the tensor

$$\sum_{\sigma \in S_n} \epsilon(\sigma) e^{\sigma(1)} \otimes e^{\sigma(2)} \otimes ... \otimes e^{\sigma(n)} \in (V^*)^{\otimes n}.$$

To each edge labeled by g we assign a tensor in $V^* \otimes V \simeq End_{\mathcal{R}}(V)$ corresponding to $j_{G,n}(g) \in SL_n(\mathcal{R}) \subset M_n(\mathcal{R})$.

Let D_0 denote a graph D decomposed into pieces. We assign to D_0 the tensor product of tensors corresponding to them. We denote this tensor by $T(D_0)$. Notice that $T(D_0) \in V^{\otimes N} \otimes (V^*)^{\otimes N}$, where $N =$ the number of 1-valent sources in D_0 $=$ the number of 1-valent sinks in D_0.

Now we glue all components of D_0 together to get the graph D back. Whenever we glue an end of one piece to a beginning of another piece in D_0, we make the corresponding contraction on $T(D_0)$. More specifically, suppose that the two free ends glued together correspond to the two underlined components:

$$T(D_0) \in V \otimes ... \otimes \underline{V} \otimes ... \otimes V \otimes V^* \otimes ... \otimes \underline{V^*} \otimes ... \otimes V^*.$$

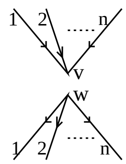
By applying to this tensor space the contraction map $\underline{V} \otimes \underline{V^*} \to R$ (which is the evaluation map $(v, f) \to f(v)$), we send $T(D_0) \in V^{\otimes N} \otimes (V^*)^{\otimes N}$ to an element of $V^{\otimes N-1} \otimes (V^*)^{\otimes N-1}$. By repeating this process until we get the graph D back, we obtain an element of $\mathcal{R}$, if $D \in \mathcal{G}_n(G)$, or an element of $M_n(\mathcal{R})$, if $D \in \mathcal{G}'_n(G)$. Notice that the above construction does not depend on the particular decomposition of D into pieces. Therefore, we have defined functions

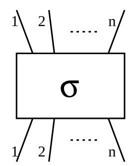
$$\Theta : \mathcal{G}'_n(G) \to M_n(\mathcal{R}), \qquad \theta : \mathcal{G}_n(G) \to \mathcal{R}. \tag{4.5}$$

Lemma 4.4. *Let D be a graph in $\mathcal{G}_n(G)$ or in $\mathcal{G}'_n(G)$. Let w be an n-valent source of D and v an n-valent sink of D. Let D_σ, for $\sigma \in S_n$, be defined as at the beginning of Section 4. If $D \in \mathcal{G}_n(G)$, then $\theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma)\theta(D_\sigma)$. If $D \in \mathcal{G}'_n(G)$, then $\Theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma)\Theta(D_\sigma)$.*

Proof. We will prove Lemma 4.4 only for $D \in \mathcal{G}_n(G)$. For $D \in \mathcal{G}'_n(G)$ the proof is identical.

Decompose D and D_σ into sources, sinks, and edges. We denote the fragment of the decomposition of D composed of the source w and the sink v by D^0. We may assume that the decomposition of D_σ is identical to that of D, except that it contains a coupon D^0_σ instead of D^0:

$D^0 =$ 

, $D^0_\sigma =$ 

.

We order the 1-valent sources and sinks of D^0_σ consistently with the ordering of the 1-valent vertices of D^0.

Let $T(D^0)$ (respectively, $T(D^0_\sigma)$) be the tensor associated to D^0 (respectively, D^0_σ). We assume that the i-th coordinate of $T(D^0) \in V^{\otimes n} \otimes V^{*\otimes n}$ corresponds to the i-th source of D^0, if $1 \le i \le n$, or to the $(i-n)$-th sink of D^0, if $n < i \le 2n$.

Recall that $\theta(D), \theta(D_\sigma) \in \mathcal{R}$ are results of contractions of tensors associated with elements of decompositions of D and D_σ. Since the decompositions of D and D_σ chosen by us differ only by elements D^0, D^0_σ, the proof of Lemma 4.4 can be reduced to a local computation on tensors. Namely, it is enough to prove that

$$T(D^0) = \sum_{\sigma \in S_n} \epsilon(\sigma) T(D^0_\sigma). \tag{4.6}$$

Notice that each edge of D^0_σ is labeled by the identity map in $End_R(V)$. This map is represented by $\sum_{i=1}^n e_i \otimes e^i$ in $V \otimes V^* \simeq End_R(V)$. Therefore, if $\sigma = id \in S_n$, then

$$T(D^0_\sigma) = \sum_{i_1, i_2, ..., i_n \in \{1,2,...,n\}} e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_n} \otimes e^{i_1} \otimes e^{i_2} \otimes ... \otimes e^{i_n}.$$

Similarly, for any $\sigma \in S_n$, we have

$$T(D^0_\sigma) = \sum_{i_1, i_2, ..., i_n \in \{1,2,...,n\}} e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_n} \otimes e^{i_{\sigma(1)}} \otimes e^{i_{\sigma(2)}} \otimes ... \otimes e^{i_{\sigma(n)}}.$$

Therefore,

$$\begin{aligned} &\sum_{\sigma \in S_n} \epsilon(\sigma) T(D^0_\sigma) \\ &\quad = \sum_{\substack{\sigma \in S_n \\ i_1, i_2, ..., i_n \in \{1,2,...,n\}}} \epsilon(\sigma) e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_n} \otimes e^{i_{\sigma(1)}} \otimes e^{i_{\sigma(2)}} \otimes ... \otimes e^{i_{\sigma(n)}}. \end{aligned} \tag{4.7}$$

Note that we can assume that the numbers $i_1, i_2, ..., i_n$ appearing on the right side of (4.7) are all different. Indeed, if $i_j = i_k, j \neq k$, then there is an equal number of even and odd permutations contributing the term

$$e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_n} \otimes e^{j_1} \otimes e^{j_2} \otimes ... \otimes e^{j_n}$$

to the sum on the right side of (4.7), for any $j_1, j_2, ..., j_n$.

Therefore, we can assume that the numbers $(i_1, i_2, ..., i_n)$ appearing in each term of the sum on the right side of (4.7) form a permutation τ of $(1, 2, ..., n)$. Hence we have

$$\sum_{\sigma\in S_n}\epsilon(\sigma)T(D^0_\sigma)$$
$$=\sum_{\sigma,\tau\in S_n}\epsilon(\sigma)e_{\tau(1)}\otimes e_{\tau(2)}\otimes ...\otimes e_{\tau(n)}\otimes e^{\tau(\sigma(1))}\otimes e^{\tau(\sigma(2))}\otimes ...\otimes e^{\tau(\sigma(n))}.$$

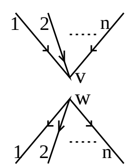
Substitute τ' for $\tau\circ\sigma$. Then $\epsilon(\sigma)=\epsilon(\tau)\epsilon(\tau')$, and we get

$$\sum_{\sigma\in S_n}\epsilon(\sigma)T(D^0_\sigma)$$
$$=\sum_{\tau,\tau'\in S_n}\epsilon(\tau)\epsilon(\tau')e_{\tau(1)}\otimes e_{\tau(2)}\otimes ...\otimes e_{\tau(n)}\otimes e^{\tau'(1)}\otimes e^{\tau'(2)}\otimes ...\otimes e^{\tau'(n)}.$$

Notice that the right side of the above equation is equal to

$$\left(\sum_{\tau\in S_n}\epsilon(\tau)e_{\tau(1)}\otimes e_{\tau(2)}\otimes ...\otimes e_{\tau(n)}\right)\otimes\left(\sum_{\tau'\in S_n}\epsilon(\tau')e^{\tau'(1)}\otimes e^{\tau'(2)}\otimes ...\otimes e^{\tau'(n)}\right).$$

But the expression above is exactly the tensor assigned to:



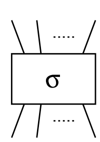
Therefore we have proved (4.6) and completed the proof of Lemma 4.4. □

The study of SL_n-actions on linear spaces was one of the main objectives of classical invariant theory. In particular, Weyl ([We]) determined all invariants of the action of $SL(V)$ on $V\otimes ...\otimes V\otimes V^*\otimes ...\otimes V^*$ and gave a full description of relations between them. The set of "typical" invariants consists of brackets $[v_1,...,v_n]=Det(v_1,...,v_n), [\phi_1,...,\phi_n]^*=Det(\phi_1,...,\phi_n)$, where $v_1,...,v_n\in V, \phi_1,...,\phi_n\in V^*$, and contractions $\phi_j(v_i)$. The identity

$$[v_1,...,v_n][\phi_1,...,\phi_n]^*=Det(\phi_j(v_i))^n_{i,j=1} \tag{4.8}$$

is one of the fundamental identities relating the typical invariants. The bracket $[\cdot,\cdot,...,\cdot]$ is a skew symmetric linear functional on $V\otimes V\otimes ...\otimes V$ and hence an element of $\bigwedge^n V^*$. Similarly, $[\cdot,\cdot,...,\cdot]^*\in\bigwedge^n V$. Note that the sources and sinks of graphs considered by us are labeled exactly by the tensors $[\cdot,\cdot,...,\cdot]^*$ and $[\cdot,\cdot,...,\cdot]$. (However, V is in our case a free module over $\mathcal{R}=Rep^R_n(G)$.)

It follows from the proof of Lemma 4.4 that

$$\sum_{\sigma\in S_n}\epsilon(\sigma)$$ 

represents the tensor in $Hom(V\otimes ...\otimes V\otimes V^*\otimes ...\otimes V^*,\mathcal{R})=V^*\otimes ...\otimes V^*\otimes V\otimes ...\otimes V$ assigning to $(v_1,v_2,...,v_n,\phi_1,\phi_2,...,\phi_n)$ the value $Det(\phi_j(v_i))^n_{i,j=1}$. Therefore, the identity

$$\theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma)\theta(D_\sigma)$$

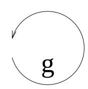
is essentially equivalent to (4.8).

Lemma 4.5. *Let L_g, E_g, EL_g be graphs defined as in Section 3 but considered as elements of $\mathcal{G}_n(G)$ and $\mathcal{G}'_n(G)$, i.e.*

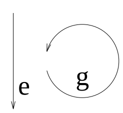
(1) *$L_g \in \mathcal{G}_n(G)$ is a single loop labeled by the conjugacy class of $g \in G$,*
(2) *$E_g \in \mathcal{G}'_n(G)$ is a single edge labeled by $g \in G$, and*
(3) *$EL_g \in \mathcal{G}'_n(G)$ is a graph composed of an edge labeled by the identity in G and of a loop labeled by the conjugacy class of $g \in G$.*

Under the above assumptions the functions Θ and θ satisfy conditions (1) and (2) of Theorem 3.6.

Proof. (1) L_g can be decomposed into a single arc



which has associated the tensor $j_{G,n}(g) \in SL_n(\mathcal{R}) \subset V^* \otimes V$. The contraction of this tensor gives $\theta(L_g) = Tr(j_{G,n}(g))$.
(2) E_g is a single arc. Therefore $\Theta(E_g) = T(E_g) = j_{G,n}(g)$.
(3) EL_g can be decomposed into:



The tensor associated with this decomposition is $id \otimes j_{G,n}(g) \in End(V) \otimes End(V)$. After making a contraction corresponding to the identification of the ends of the arc, we get

$$\Theta(EL_g) = id \cdot Tr(j_{G,n}(g)) \in End(V).$$

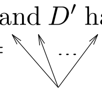
□

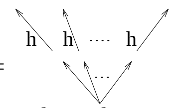
Lemma 4.6. *Let $D, D' \in \mathcal{G}_n(G)$ be two graphs which are identical as unlabeled graphs and which have the same labeling of edges and loops except the labeling of edges incident to a vertex v. Moreover, suppose that*

(1) *if v is a source, then the edges in D incident to v are labeled by $g_1, g_2, ..., g_n$ and the edges in D' incident to v are labeled by $g_1h, g_2h, ..., g_nh$ for some $g_1, g_2, ..., g_n, h \in G$;*
(2) *if v is a sink, then the edges in D incident to v are labeled by $g_1, g_2, ..., g_n$ and the edges in D' incident to v are labeled by $hg_1, hg_2, ..., hg_n$ for some $g_1, g_2, ..., g_n, h \in G$.*

Under the above assumptions $\theta(D) = \theta(D')$. An analogous fact is true for graphs in $\mathcal{G}'_n(G)$.

Proof. We prove part (1) only. The proof of part (2) is analogous.

Let v be a source. Notice that D and D' have identical decompositions into sinks, sources, and arcs except that $D_0 =$  is an element of the decomposition of

D and the diagram $D'_0 =$ 

is a fragment of a decomposition of D'.

Therefore we need to show that the tensors assigned to the above diagrams are identical. Notice that the tensor associated to D_0, $T(D_0)$, is an element of the one-dimensional $\mathcal{R}$-linear space of skew-symmetric tensors $\bigwedge^n V \subset V^n$. Let $A : V \to V$ be an endomorphism given in the standard coordinates of V by $j_{G,n}(h) \in SL_n(\mathcal{R})$. A induces an endomorphism $\wedge A : \bigwedge^n V \to \bigwedge^n V$ with the property that $\wedge A(T(D_0)) = Det(A)T(D_0) \in \bigwedge^n V$. Notice that $\wedge A(T(D_0))$ is exactly the tensor associated to D'_0. Since $Det(A) = 1$, the tensors associated to D_0 and D'_0 are equal. □

Proof of Theorem 3.6. Let us extend the functions θ and Θ to all R-linear combinations of graphs in $\mathcal{G}_n(G)$ and $\mathcal{G}'_n(G)$ respectively. Facts 4.1 and 4.2 and Lemmas 4.4, 4.5, and 4.6 imply that these functions descend to R-linear homomorphisms

$$\Theta : \mathbb{A}_n(X, x_0) \to M_n(\mathcal{R}), \qquad \theta : \mathbb{A}_n(X) \to \mathcal{R}.$$

By Lemma 4.5, Θ and θ satisfy conditions (1) and (2) of Theorem 3.6.

We have showed in Proposition 3.5(4) that $\mathbb{A}_n(X, x_0)$ is generated by elements E_γ and $EL_{\gamma'}$, for $\gamma, \gamma' \in G = \pi_1(X, x_0)$. By Proposition 2.4 and the paragraph preceding it, $\Theta(E_\gamma) = j_{G,n}(\gamma)$ and $\Theta(EL_{\gamma'}) = Tr(j_{G,n}(\gamma'))$ belong to

$$M_n(Rep_n^R(G))^{GL_n(R)}.$$

Therefore the image of Θ lies in $M_n(Rep_n^R(G))^{GL_n(R)}$. We show analogously that the image of θ lies in $Rep_n^R(G)^{GL_n(R)}$. Therefore the proof will be completed if we show that the diagram of Theorem 3.6 commutes.

Let $D \in \mathcal{G}'_n(G)$, $G = \pi_1(X, x_0)$, represent an element of $\mathbb{A}_n(X, x_0)$. Then $\Theta(D) \in M_n(Rep_n^R(G))^{GL_n(R)}$ is the result of a contraction of tensors associated with elements of some decomposition of D. Notice that $\mathbb{T}(D)$ is an element of $\mathbb{A}_n(X)$ represented by the diagram D with its 1-valent vertices identified.[2] Hence $\theta(\mathbb{T}(D))$ is a contraction of $\Theta(D)$, i.e. $\theta(\mathbb{T}(D)) = Tr(\Theta(D))$. Since the elements $D \in \mathcal{G}'_n(G)$ span $\mathbb{A}_n(X, x_0)$, the diagram of Theorem 3.6 commutes. □

5. Proof of Theorem 3.7

Now we assume that R is a field of characteristic 0.

We start by stating the first and second fundamental theorems of invariant theory, following the approach of Procesi, [Pro-1] (compare also [Ra]).

Let I be an infinite set. Let $P_n(I)$ and A_i be as before,

$$P_n(I) = R[x^i_{jk},\ j, k \in \{1, 2, ..., n\}, i \in I], \qquad A_i = (x^i_{jk}) \in M_n(P_n(I)).$$

We are going to present Procesi's description of the ring $M_n(P_n(I))^{GL_n(R)}$.

Let T be a commutative R-algebra freely generated by the symbols

$$Tr(X_{i_1} X_{i_2} ... X_{i_k}),$$

where $i_1, i_2, ..., i_k \in I$. We adopt the convention that $Tr(M) = Tr(N)$ if and only if the monomial N is obtained from M by a cyclic permutation. Let $T\{X_i\}_{i \in I}$ be a non-commutative T-algebra freely generated by variables X_i, $i \in I$. We

[2]Recall that $\mathbb{T}$ was defined in the paragraph preceding Fact 3.4.

have a natural T-linear homomorphism $Tr : T\{X_i\}_{i\in I} \to T$ which assigns to $X_{i_1}X_{i_2}...X_{i_k} \in T\{X_i\}_{i\in I}$ an element $Tr(X_{i_1}X_{i_2}...X_{i_k}) \in T$.

There is a homomorphism $\pi : T\{X_i\}_{i\in I} \to M_n(P_n(I))$ uniquely determined by the conditions:

- $\pi(X_i) = A_i$,
- $\pi(Tr(X_{i_1}X_{i_2}...X_{i_k})) = Tr(A_{i_1}A_{i_2}...A_{i_k}) \in P_n(I) \subset M_n(P_n(I))$.[3]

Proposition 2.3 and Lemma 2.2 imply that the image of π is fixed by the $GL_n(R)$-action on $M_n(P_n(I))$, i.e. $\pi : T\{X_i\}_{i\in I} \to M_n(P_n(I))^{GL_n(R)}$.

Notice that the following diagram commutes:

$$\begin{array}{ccc} T\{X_i\}_{i\in I} & \xrightarrow{\pi} & M_n(P_n(I))^{GL_n(R)} \\ \downarrow{\scriptstyle Tr} & & \downarrow{\scriptstyle Tr} \\ T & \xrightarrow{\pi_{|T}} & P_n(I)^{GL_n(R)} \end{array}$$

The following version of *The First Fundamental Theorem* of invariant theory of $n \times n$ matrices is due to Procesi, [Pro-1].

Theorem 5.1. *$\pi : T\{X_i\}_{i\in I} \to M_n(P_n(I))^{GL_n(R)}$ is an epimorphism.*

Before we state the second fundamental theorem of invariant theory of $n \times n$ matrices, we need some preparations.

Suppose that $\{1, 2, ..., m\} \subset I$ and specify $i_0 \in \{1, 2, ..., m\}$. We can present any $\sigma \in S_m$ as a product of cycles in such a way that i_0 is the first element of the first cycle, $\sigma = (i_0, i_1, ..., i_s)(j_0, j_1, ..., j_t)...(k_0, k_1, ..., k_v)$. We define $\Phi_{\sigma,i_0}(X_1, X_2, ..., X_m)$ to be equal to

$$X_{i_0}X_{i_1}...X_{i_s}Tr(X_{j_0}X_{j_1}...X_{j_t})...Tr(X_{k_0}X_{k_1}...X_{k_v}) \in T\{X_i\}_{i\in I}.$$

We also define another expression which does not depend on i_0:

$$\Phi_\sigma(X_1, X_2, ..., X_m) = Tr(X_{i_0}X_{i_1}...X_{i_s})Tr(X_{j_0}X_{j_1}...X_{j_t})...Tr(X_{k_0}X_{k_1}...X_{k_v}) \in T.$$

Let

$$F(X_1, X_2, ..., X_m) = \sum_{\sigma\in S_m} \epsilon(\sigma)\Phi_\sigma(X_1, X_2, ..., X_m) \in T.$$

$F(X_1, X_2, ..., X_{n+1})$ is called the fundamental trace identity of $n \times n$ matrices.

Procesi argues that there exists a unique element $G(X_1, X_2, ..., X_n) \in T\{X_i\}_{i\in I}$, involving only the variables $X_1, ..., X_n$ and the traces of monomials in these variables, such that

$$F(X_1, X_2, ..., X_{n+1}) = Tr(G(X_1, X_2, ..., X_n)X_{n+1}) \in T\{X_i\}_{i\in I}.$$

Procesi gives an explicit formula for $G(X_1, X_2, ..., X_n)$, but we want to give a different formula, which will be more suitable for our purposes.

Lemma 5.2.

$$G(X_1, X_2, ..., X_n) = \sum_{\sigma\in S_n} \epsilon(\sigma)\Phi_\sigma(X_1, X_2, ..., X_n) - \sum_{\substack{i\in\{1,2,...,n\} \\ \sigma\in S_n}} \epsilon(\sigma)\Phi_{\sigma,i}(X_1, X_2, ..., X_n).$$

[3] We identify $P_n(I)$ with the scalar matrices in $M_n(P_n(I))$.

Proof. It follows from the remarks preceding Lemma 5.2 that it is enough to show that if we multiply the right side of the equation of Lemma 5.2 by X_{n+1}, then the trace of it will be equal to $F(X_1, ..., X_{n+1})$, i.e. we have to prove that

(5.1)
$$\sum_{\sigma\in S_n} \epsilon(\sigma)\Phi_\sigma(X_1, X_2, ..., X_n)Tr(X_{n+1}) - \sum_{\substack{i\in\{1,2,...,n\}\\ \sigma\in S_n}} \epsilon(\sigma)Tr(\Phi_{\sigma,i}(X_1, X_2, ..., X_n)X_{n+1}) = F(X_1, X_2, ..., X_{n+1}).$$

Notice that $\epsilon(\sigma)\Phi_\sigma(X_1, X_2, ..., X_n)Tr(X_{n+1}) = \epsilon(\sigma')\Phi_{\sigma'}(X_1, X_2, ..., X_n, X_{n+1})$, where $\sigma' \in S_{n+1},\ \sigma'(i) = \sigma(i)$, for $i \in \{1, 2, ..., n\}$, and $\sigma'(n + 1) = n + 1$.

Similarly we can simplify $Tr(\Phi_{\sigma,i}(X_1, X_2, ..., X_n)X_{n+1})$. Suppose that

$$\sigma = (i_0, i_1, ..., i_s)(j_0, j_1, ..., j_t)...(k_0, k_1, ..., k_v) \in S_n,$$

where $i_0 = i$. Then

$$Tr(\Phi_{\sigma,i}(X_1, X_2, ..., X_n)X_{n+1}) = \Phi_{\sigma'}(X_1, X_2, ..., X_n, X_{n+1}),$$

for $\sigma' = (i_0, i_1, ..., i_s, n+1)(j_0, j_1, ..., j_t)...(k_0, k_1, ..., k_v) \in S_{n+1}$. Notice that $\epsilon(\sigma') = -\epsilon(\sigma)$. Therefore the left side of the equation (5.1) is equal to

$$\sum_{\substack{\sigma'\in S_{n+1},\\ \sigma'(n+1)=n+1}} \epsilon(\sigma')\Phi_\sigma(X_1, X_2, ..., X_n, X_{n+1}) + \sum_{\substack{i\in\{1,2,...,n\},\ \sigma'\in S_{n+1}\\ \text{such that } \sigma'(n+1)=i}} \epsilon(\sigma')\Phi_\sigma(X_1, X_2, ..., X_n, X_{n+1}).$$

The above expression is obviously equal to $F(X_1, X_2, ..., X_{n+1})$. □

Now we are ready to state *The Second Fundamental Theorem* of invariant theory of $n \times n$ matrices, [Pro-1].

Theorem 5.3. *The kernel of π is generated by elements $G(M_1, M_2, ..., M_n)$ and $F(N_1, N_2, ..., N_{n+1})$, where $M_1, M_2, ..., M_n, N_1, N_2, ..., N_{n+1}$ are all possible monomials in the variables $X_i,\ i \in I$.*

Let X be a path-connected topological space. We choose a presentation $\langle g_i, i \in I | r_j, j \in J\rangle$ of $G = \pi_1(X, x_0)$ such that

- I is an infinite set,
- the inverse of every generator is also a generator, and
- the defining relations r_j are products of non-negative powers of generators.

Note that such a presentation always exists (even if G is finitely generated).

Let $\psi : T\{X_i\}_{i\in I} \to \mathbb{A}_n(X, x_0)$ be an R-homomorphism such that $\psi(X_i) = E_{g_i}$ and $\psi(Tr(X_{i_1}X_{i_2}...X_{i_k})) = EL_{g_{i_1}g_{i_2}...g_{i_k}}$. Recall that, by Proposition 3.4, $\mathbb{A}_n(X)$ can be considered as a subalgebra of $\mathbb{A}_n(X, x_0)$ in such a way that $L_\gamma \in \mathbb{A}_n(X)$ is identified with $EL_\gamma \in \mathbb{A}_n(X, x_0)$. Hence $\psi(Tr(X_{i_1}X_{i_2}...X_{i_k})) \in \mathbb{A}_n(X)$ and ψ restricts to $\psi : T \to \mathbb{A}_n(X)$. Moreover, the following diagram commutes:

(5.2)
$$\begin{array}{ccc} T\{X_i\}_{i\in I} & \xrightarrow{\psi} & \mathbb{A}_n(X, x_0) \\ \downarrow{\scriptstyle Tr} & & \downarrow{\scriptstyle Tr} \\ T & \xrightarrow{\psi} & \mathbb{A}_n(X) \end{array}$$

We are going to show that the kernel of $\psi : T\{X_i\}_{i\in I} \to \mathbb{A}_n(X, x_0)$ contains the kernel of $\pi : T\{X_i\}_{i\in I} \to M_n(P_n(I))^{GL_n(R)}$ and therefore ψ descends to a homomorphism $M_n(P_n(I))^{GL_n(R)} \to \mathbb{A}_n(X, x_0)$.

We will need the following fact, due to Formanek (Proposition 45 [For]).

Proposition 5.4. *For any matrix $A \in M_n(R)$*

$$Det(A) = \frac{1}{n!} \sum_{\sigma\in S_n} \epsilon(\sigma) Tr(A^{c_1}) Tr(A^{c_2}) ... Tr(A^{c_k}),$$

where $c_1, c_2, ..., c_k$ denote the lengths of all cycles in σ.

For completeness we sketch a proof of Proposition 5.4. A multilinearization of the determinant, $Det : M_n(R) \to R$, gives a function on n-tuples of $n \times n$ matrices,

$$\mathcal{M}(X_1, ..., X_n) = \sum_{\sigma\in S_n} Det(X_\sigma),$$

where X_σ is a matrix whose i-th row is the i-th row of $X_{\sigma(i)}$. Note that $\mathcal{M}(A, ..., A) = n!Det(A)$, and therefore the identity of Proposition 5.4 is a special case of the following identity:

$$\mathcal{M}(X_1, ..., X_n) = \sum_{\sigma\in S_n} \epsilon(\sigma)\Phi_\sigma(X_1, ..., X_n),$$

where Φ_σ was defined in the second paragraph after Theorem 5.1. Formanek gives the following proof of the above identity. Assume that $1, 2, ..., n \in I$. Since $A_1, ..., A_n \in M_n(P_n(I))$ represent generic matrices, in order to prove the above identity it is enough to show it for $X_1 = A_1, ..., X_n = A_n$. Since $\mathcal{M}(A_1, ..., A_n)$ is an invariant polynomial function on n-tuples of matrices, the First Fundamental Theorem implies that $\mathcal{M}(A_1, ..., A_n)$ can be expressed in terms of traces of monomials in $A_1, ..., A_n$. Since $\mathcal{M}(A_1, ..., A_n)$ is linear with respect to $A_1, ..., A_n$, it is a linear combination of terms $Tr(A_{i_1} ... A_{i_s}) ... Tr(A_{j_1} ... A_{j_t})$, where $i_1, ... i_s, ..., j_1, ..., j_t$ form a permutation of $1, 2, ..., n$. Therefore

$$\mathcal{M}(A_1, ..., A_n) = \sum_{\sigma\in S_n} \alpha_\sigma \Phi_\sigma(A_1, ..., A_n),$$

and hence

$$\mathcal{M}(X_1, ..., X_n) = \sum_{\sigma\in S_n} \alpha_\sigma \Phi_\sigma(X_1, ..., X_n), \tag{5.3}$$

for any $n \times n$ matrices $X_1, ..., X_n$. We need to prove that $\alpha_\sigma = \epsilon(\sigma)$. If we restrict the above equation to matrices $A_1, ..., A_n \in M_{n-1}(P_{n-1}(I))$ embedded into $M_n(P_{n-1}(I))$ in the standard, non-unit-preserving way, we will get the following polynomial identity on $(n-1) \times (n-1)$ matrices:

$$\sum_{\sigma\in S_n} \alpha_\sigma \Phi(A_1, ..., A_n) = 0.$$

It is not difficult to see that the Second Fundamental Theorem implies that $F(A_1, ..., A_n)$ is the only (up to scalar) n-linear trace identity of degree n on matrices $A_1, ..., A_n \in M_{n-1}(P_{n-1}(I))$. Therefore $\alpha_\sigma = \epsilon(\sigma)c$, for some fixed c. Substituting the matrix (x_{ij}) with a single non-zero entry $x_{ii} = 1$ for X_i in (5.3), we get $c = 1$. Thus the proof of Proposition 5.4 is finished.

The specialization $A = Id \in M_n(R)$ in Proposition 5.4 yields the following corollary.

Corollary 5.5. *For any positive integer n,*

$$\sum_{\sigma\in S_n}\epsilon(\sigma)n^{c(\sigma)}=n!,$$

where $c(\sigma)$ is the number of cycles in the cycle decomposition of σ.

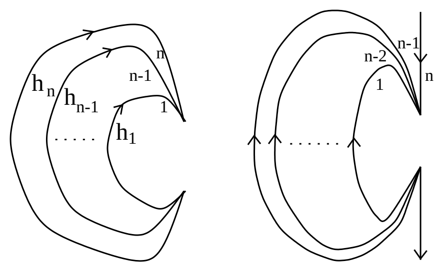
From the definition of $T\{X_i\}_{i\in I}$ it immediately follows that for any family of matrices $\{M_i\}_{i\in I}\subset M_n(R)$ there is a well-defined substitution

$$X_i\to M_i,\qquad Tr(X_{i_1}X_{i_2}...X_{i_k})\to Tr(M_{i_1}M_{i_2}...M_{i_k})\in R\subset M_n(R),$$

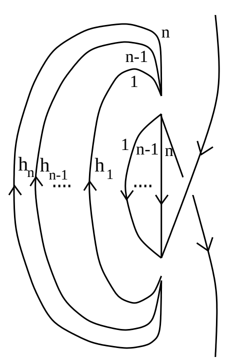
which can be extended to the whole ring $T\{X_i\}_{i\in I}$. Therefore, if $H(X_{i_1},...,X_{i_k})$ is an element of $T\{X_i\}_{i\in I}$ involving variables $X_{i_1},...,X_{i_k}$, then $H(M_{i_1},...,M_{i_k})$ is a well-defined matrix in $M_n(R)$.

Lemma 5.6. *If $N_1,N_2,...,N_n$ are any monomials in the variables $X_i,\ i\in I$, then $\psi(G(N_1,N_2,...,N_n))=0$.*

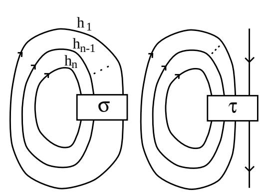
Proof. By the definition of ψ (given in the second paragraph after Theorem 5.3), $\psi(N_i)=E_{h_i}$, for some $h_1,h_2,...,h_n\in G$. Consider the following graph D in $\mathcal{G}'_n(G)$:



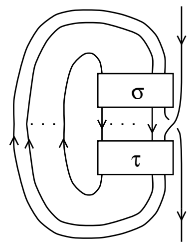
in which we omitted labels of edges labeled by the identity in G. Notice that D can also be presented in the following way:



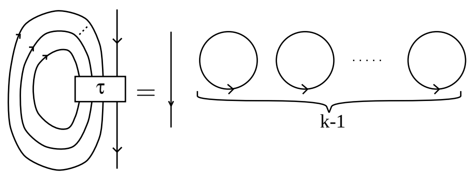
Since the vertices of D can be resolved in two possible ways (corresponding to the two diagrams above), we obtain the following equation:

(5.4) $$\sum_{\sigma,\tau\in S_n}\epsilon(\sigma)\epsilon(\tau)$$  $$=\sum_{\sigma,\tau\in S_n}\epsilon(\sigma)\epsilon(\tau)D_{\sigma,\tau},$$

where $D_{\sigma,\tau}$ is a graph of the form:

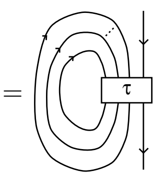


If $\tau \in S_n$ decomposes into $k = c(\tau)$ cycles, then



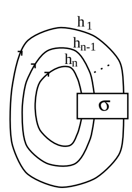
.

Therefore, by Corollary 5.5,

$$\sum_{\tau \in S_n} \epsilon(\tau) =$$


$$= \sum_{\tau \in S_n} \epsilon(\tau) n^{c(\tau)-1} = (n-1)!.$$

Notice moreover that



is equal to $\psi(\Phi_\sigma(N_1, N_2, ..., N_n))$. Therefore the left side of (5.4) is equal to the value of ψ on

$$(n-1)! \sum_{\sigma \in S_n} \epsilon(\sigma) \Phi_\sigma(N_1, N_2, ..., N_n).$$

Now we are going to consider the right side of (5.4). Notice that the single arc in $D_{\sigma,\tau}$ is labeled by an element $h_{i_s} ... h_{i_1} h_{i_0} \in G$, where $i_0 = \tau(n), i_1 = \tau\sigma(i_0), ..., i_s = \tau\sigma(i_{s-1})$, and $\sigma(i_s) = n$. Since $\tau(\sigma(i_s)) = \tau(n) = i_0$, $(i_s, i_{s-1}, ..., i_1, i_0)$ is a cycle of the permutation $(\tau\sigma)^{-1} \in S_n$.

Note that every loop in $D_{\sigma,\tau}$ is labeled by the conjugacy class of $h_{j_t} h_{j_{t-1}} ... h_{j_1} h_{j_0}$, where $(j_t, j_{t-1}, ..., j_1, j_0)$ is a cycle of $(\tau\sigma)^{-1} \in S_n$ disjoint from $(i_s, i_{s-1}, ..., i_1, i_0)$. Therefore $D_{\sigma,\tau}$ is the value of ψ on

$$N_{i_s} ... N_{i_1} N_{i_0} Tr(N_{j_t} ... N_{j_1} N_{j_0}) ... Tr(N_{k_v} ... N_{k_1} N_{k_0}),$$

where

$$(i_s, ..., i_1, i_0)(j_t, ..., j_1, j_0) ... (k_v, ..., k_1, k_0)$$

is the cycle decomposition of $(\tau\sigma)^{-1}$. The above expression is equal to

$$\Phi_{(\tau\sigma)^{-1},i_s}(N_1,N_2,...,N_n)=\Phi_{(\tau\sigma)^{-1},\sigma^{-1}(n)}(N_1,N_2,...,N_n).$$

Therefore, the right side of (5.4) is the value of ψ on

$$\sum_{\sigma,\tau\in S_n}\epsilon(\sigma)\epsilon(\tau)\Phi_{(\tau\sigma)^{-1},\ \sigma^{-1}(n)}(N_1,N_2,...,N_n)\in T\{X_i\}_{i\in I}.$$

Let us replace $(\tau\sigma)^{-1}$ by κ in the expression above. Then we get

$$\sum_{\sigma,\kappa\in S_n}\epsilon(\kappa)\Phi_{\kappa,\sigma^{-1}(n)}(N_1,N_2,...,N_n)=(n-1)!\sum_{\kappa\in S_n,\ i\in\{1,2,...,n\}}\epsilon(\kappa)\Phi_{\kappa,i}(N_1,N_2,...,N_n).$$

After comparing the above algebraic descriptions of the two sides of (5.4) we see that for any monomials $N_1,N_2,...,N_n$ the following element of $T\{X_i\}_{i\in I}$ belongs to $Ker\ \psi$:

$$\sum_{\sigma\in S_n}\epsilon(\sigma)\Phi_\sigma(N_1,N_2,...,N_n)-\sum_{\kappa\in S_n,\ i\in\{1,2,...,n\}}\epsilon(\kappa)\Phi_{\kappa,i}(N_1,N_2,...,N_n).$$

By Lemma 5.2 the above expression is equal to $G(N_1,N_2,...,N_n)$. Therefore $\psi(G(N_1,N_2,...,N_n))=0$. □

Lemma 5.7. *Let $N_1,N_2,...,N_{n+1}$ be any monomials in the variables $X_i,\ i\in I$. Then $\psi(F(N_1,N_2,...,N_{n+1}))=0$.*

Proof. By definition, $F(N_1,N_2,...,N_{n+1})=Tr(G(N_1,N_2,...,N_n)N_{n+1})$. By (5.2), ψ commutes with the trace function. Therefore

$$\psi(F(N_1,N_2,...,N_{n+1}))=\psi(Tr(G(N_1,N_2,...,N_n)N_{n+1}))$$
$$=Tr(\psi(G(N_1,N_2,...,N_n))\psi(N_{n+1}))=0.$$

□

Lemmas 5.6 and 5.7 and the Second Fundamental Theorem imply that the kernel of $\psi:T\{X_i\}_{i\in I}\to\mathbb{A}_n(X,x_0)$ contains the kernel of $\pi:T\{X_i\}_{i\in I}\to M_n(P_n(I))^{GL_n(R)}$. Therefore we have the following corollary.

Corollary 5.8. *There exists an R-algebra homomorphism $\psi':M_n(P_n(I))^{GL_n(R)}\to\mathbb{A}_n(X,x_0)$ such that $\psi'(A_i)=E_{g_i}$ and $\psi'(Tr(A_{i_1}A_{i_2}...A_{i_k}))=EL_{g_{i_1}g_{i_2}...g_{i_k}}$, for any $i_1,i_2,...,i_k\in I$.*

The epimorphism $\eta:P_n(I)\to Rep_n^R(G)$ introduced in Section 2 induces an epimorphism $M_n(\eta):M_n(P_n(I))\to M_n(Rep_n^R(G))$ and, therefore, by restriction, a homomorphism $M_n(\eta)^{GL_n(R)}:M_n(P_n(I))^{GL_n(R)}\to M_n(Rep_n^R(G))^{GL_n(R)}$. Our goal is to show that ψ' descends to

$$\psi'':M_n(Rep_n^R(G))^{GL_n(R)}\to\mathbb{A}_n(X,x_0)$$

such that the following diagram commutes:

(5.5)
$$\begin{array}{ccc} M_n(P_n(I))^{GL_n(R)} & & \\ \downarrow{\scriptstyle M_n(\eta)^{GL_n(R)}} & \overset{\psi'}{\searrow} & \\ M_n(Rep_n^R(G))^{GL_n(R)} & \xrightarrow{\psi''} & \mathbb{A}_n(X,x_0) \end{array}$$

In order to prove this fact we need to show that $Ker\ M_n(\eta)^{GL_n(R)} \subset Ker\ \psi'$. We will use the following lemma.

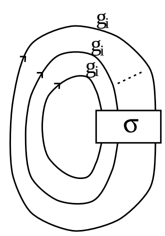
Lemma 5.9. (1) $Det(A_i) \in P_n(I)^{GL_n(R)} \subset M_n(P_n(I))^{GL_n(R)}$.
(2) $\psi'(Det(A_i)) = 1$, *for any* $i \in I$.

Proof. (1) By Proposition 5.4, $Det(A_i)$ can be expressed as a linear combination of traces of powers of A_i. By Lemma 2.2(2), $A_i^k \in M_n(P_n(I))^{GL_n(R)}$, and hence, by Proposition 2.3, $Tr(A_i^k) \in P_n(I)^{GL_n(R)}$. Finally, by Lemma 2.2(1) there is a natural embedding $P_n(I)^{GL_n(R)} \subset M_n(P_n(I))^{GL_n(R)}$.

(2) If $c_1, c_2, ..., c_k$ are the lengths of all cycles of $\sigma \in S_n$, then ψ' maps

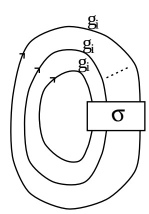
$$Tr(A_i^{c_1})Tr(A_i^{c_2})...Tr(A_i^{c_k})$$

to a graph

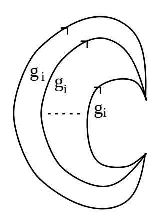


Therefore, by Proposition 5.4 and Fact 4.1,

$$\psi'(Det(A_i)) = \frac{1}{n!}\sum_{\sigma\in S_n}\epsilon(\sigma)$$



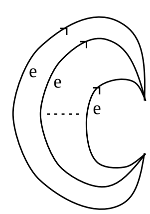
$$= \frac{1}{n!}$$



.

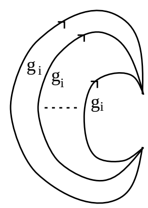
Analogously,

$$1 = \psi'(Det(\mathbf{1})) = \frac{1}{n!}$$

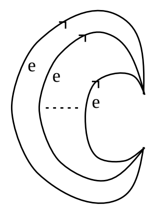


,

where e is the identity in G. But, by (4.3) (or, equivalently, (4.4)),



$=$



.

Therefore $\psi'(Det(A_i)) = 1$. ∎

The next proposition is due to Procesi. Since the proof of this proposition is hidden in the proof of Theorem 2.6 in [Pro-2], we will recall it here for completeness of this paper.[4]

Proposition 5.10. *Let $\mathcal{J} \triangleleft M_n(P_n(I))^{GL_n(R)}$ be a two-sided ideal and let $\mathcal{J}'$ be the ideal in $P_n(I)$ generated by the entries of elements of $M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \triangleleft M_n(P_n(I))$. Then:*

1. *$M_n(P_n(I))\mathcal{J}M_n(P_n(I)) = M_n(\mathcal{J}') \triangleleft M_n(P_n(I))$.*
2. *There is a unique $GL_n(R)$-action on $M_n(P_n(I)/\mathcal{J}')$ such that the natural projection $i : M_n(P_n(I)) \to M_n(P_n(I)/\mathcal{J}')$ is $GL_n(R)$-equivariant.*
3. *i induces a homomorphism*

$$j : M_n(P_n(I))^{GL_n(R)}/\mathcal{J} \to M_n(P_n(I)/\mathcal{J}')^{GL_n(R)}$$

which is an isomorphism of R-algebras.

Proof. (1) This follows from the basic algebraic fact that every ideal $\mathcal{I}$ in $M_n(R)$, for any ring R with 1, is of the form $M_n(\mathcal{I}')$, where $\mathcal{I}'$ is the ideal in R generated by the entries of a generating set of the ideal $\mathcal{I}$.

(2) Let $B \in GL_n(R)$, and let $B*$ denote the action of B on $M_n(P_n(I))$. B leaves $M_n(\mathcal{J}')$ invariant. Indeed, any element $C \in M_n(\mathcal{J}')$ is of the form $\sum_i M_iC_iN_i$, where $M_i, N_i \in M_n(P_n(I))$, $C_i \in \mathcal{J}$, and therefore

$$B * C = \sum_i (B * M_i)(B * C_i)(B * N_i) \in M_n(\mathcal{J}').$$

This implies that the action of $GL_n(R)$ on $M_n(P_n(I)/\mathcal{J}')$ is well defined. All other statements of (2) are obvious consequences of this fact.

(3) For any rational action of $GL_n(R)$ on any R-vector space N there exists a linear projection $\nabla : N \to N^{GL_n(R)}$, called the Reynolds operator, with the following properties:

- (i) $\nabla(x) = x$ for $x \in N^{GL_n(R)}$, and, therefore, ∇ is an epimorphism.
- (ii) ∇ is natural with respect to $GL_n(R)$-equivariant maps $N \to N'$.
- (iii) If N is an algebra, then $\nabla(xy) = x\nabla(y)$ and $\nabla(yx) = \nabla(y)x$ for $x \in N^{GL_n(R)}$ and $y \in N$.

For more information about this operator, see [MFK] or a more elementary text [Fog].

The homomorphism i restricted to $M_n(P_n(I))^{GL_n(R)}$ induces a homomorphism

$$j : M_n(P_n(I))^{GL_n(R)}/\mathcal{J} \to M_n(P_n(I)/\mathcal{J}')^{GL_n(R)}.$$

By Property (i) of ∇, j is an epimorphism. It remains to prove that j is injective.

Choose $i_0 \in I$. For any monomial m in $P_n(I) = R[x_{jk}^i,\ i \in I, j,k = 1,2,...,n]$ we define the degree of m to be the number of appearances of the variables $x_{jk}^{i_0}$, $j,k \in \{1,2,...,n\}$, in m. This induces a grading on $P_n(I)$. We can extend this grading on $M_n(P_n(I))$ as follows. For any matrix $A = (a_{jk}) \in M_n(P_n(I))$ with a single non-zero entry a_{st}, $deg(A) = deg(a_{st})$. Note that the degree of the matrix $A_{i_0} = (x_{jk}^{i_0}) \in M_n(P_n(I))$ considered in Section 2 is 1.

[4] Compare also Proposition 9.5 in [B-H].

Let $B \in GL_n(R)$. By the definition of the $GL_n(R)$-action on $M_n(P_n(I))$ and by Lemma 2.2(2),

$$B \begin{pmatrix} B * x_{11}^{i_0} & B * x_{12}^{i_0} & \dots & B * x_{1n}^{i_0} \\ \vdots & \vdots & \dots & \vdots \\ B * x_{n1}^{i_0} & B * x_{n2}^{i_0} & \dots & B * x_{nn}^{i_0} \end{pmatrix} B^{-1} = \begin{pmatrix} x_{11}^{i_0} & x_{12}^{i_0} & \dots & x_{1n}^{i_0} \\ \vdots & \vdots & \dots & \vdots \\ x_{n1}^{i_0} & x_{n2}^{i_0} & \dots & x_{nn}^{i_0} \end{pmatrix}.$$

Therefore $B * x_{jk}^{i_0}$ is a linear combination of the variables $x_{j'k'}^{i_0}$, $j', k' = 1, 2, ..., n$, and hence the action of $GL_n(R)$ preserves the grading of $P_n(I)$. For any $M \in M_n(P_n(I))$, $B * M$ is a matrix obtained by applying the action of B to all entries of M and then conjugating the resulting matrix by B. Therefore the action of $GL_n(R)$ also preserves the grading of $M_n(P_n(I))$. The naturality of the Reynolds operators $\nabla : P_n(I) \to P_n(I)^{GL_n(R)}$ and $\nabla : M_n(P_n(I)) \to M_n(P_n(I))^{GL_n(R)}$ implies that they also preserve the gradings. This fact will be an important element of the proof of Proposition 5.10(3).

We need to show that

$$\mathcal{J} = M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \cap M_n(P_n(I))^{GL_n(R)}.$$

However, it is sufficient to show that

$$\mathcal{J} \supset M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \cap M_n(P_n(I))^{GL_n(R)},$$

since the opposite inclusion is obvious.

Let $c = \sum_i a_i c_i b_i \in M_n(P_n(I))^{GL_n(R)}$, where $a_i, b_i \in M_n(P_n(I)), c_i \in \mathcal{J}$. We will show that $c \in \mathcal{J}$. Since c involves only finitely many variables x_{jk}^i and I is infinite, we can choose $i_0 \in I$ such that $x_{jk}^{i_0}$, $j, k = 1, 2, ..., n$, do not appear in a_i, b_i, c_i. Thus $deg\ a_i = deg\ b_i = deg\ c_i = 0$.

Consider $Tr(cA_{i_0})$. By our assumptions about c and by Lemma 2.2(2), $cA_{i_0} \in M_n(P_n(I))^{GL_n(R)}$. Proposition 2.3 states that $Tr : M_n(P_n(I)) \to P_n(I)$ is $GL_n(R)$-equivariant, and therefore $Tr(cA_{i_0}) \in M_n(P_n(I))^{GL_n(R)}$. Thus

$$Tr(cA_{i_0}) = Tr\left(\nabla(cA_{i_0})\right) = Tr\left(\sum_i \nabla(a_i c_i b_i A_{i_0})\right)$$

$$= Tr\left(\sum_i \nabla(b_i A_{i_0} a_i c_i)\right) = Tr\left(\sum_i \nabla(b_i A_{i_0} a_i) c_i\right).$$

Note that $b_i A_{i_0} a_i$ has degree 1 and, since ∇ preserves the grading, $\nabla(b_i A_{i_0} a_i)$ is also of degree 1. By The First Fundamental Theorem of Invariant Theory (Theorem 5.1), $M_n(P_n(I))^{GL_n(R)}$ is generated by the elements A_i and $Tr(M)$, where M varies over the set of monomials composed of non-negative powers of matrices $A_i, i \in I$. By our definition of degree,

$$deg(A_i) = \begin{cases} 1 & \text{if } i = i_0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$deg(Tr(M)) = \text{ number of appearances of } A_{i_0} \text{ in } M.$$

Therefore, $\nabla(b_i A_{i_0} a_i)$ can be presented as

$$\sum_j p_{ij} A_{i_0} q_{ij} + \sum_k Tr(s_{ik} A_{i_0}) t_{ik},$$

for some elements $p_{ij}, q_{ij}, s_{ik}, t_{ik} \in M_n(P_n(I))^{GL_n(R)}$ of degree 0. Thus

$$Tr(cA_{i_0}) = Tr\left(\sum_i \sum_j p_{ij} A_{i_0} q_{ij} c_i + \sum_i \sum_k Tr(s_{ik} A_{i_0}) t_{ik} c_i\right)$$

$$= Tr\left(\sum_i \sum_j p_{ij} A_{i_0} q_{ij} c_i\right) + \sum_i \sum_k Tr(s_{ik} A_{i_0}) Tr(t_{ik} c_i)$$

$$= Tr\left(\left(\sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik}\right) A_{i_0}\right).$$

Therefore

$$Tr\left(\left[c - \left(\sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik}\right)\right] A_{i_0}\right) = 0$$

in $M_n(P_n(I))$. The expression in brackets above has degree 0. Note that if $deg\ d = 0,\ d \in M_n(P_n(I))$, then $Tr(dA_{i_0}) = 0$ if and only if $d = 0$. Therefore

$$c = \sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik},$$

and hence $c \in \mathcal{J}$. This completes the proof of Proposition 5.10. □

Let $\mathcal{J} \triangleleft M_n(P_n(I))^{GL_n(R)}$ be the ideal generated by elements $Det(A_i) - 1,\ i \in I$, and elements $A_{i_1} A_{i_2} ... A_{i_k} - 1$ corresponding to all defining relations $r_j = g_{i_1} g_{i_2} ... g_{i_k}$ of G. By Lemma 5.9(1) and Lemma 2.2(2), $Det(A_i) - 1$ and $A_{i_1} A_{i_2} ... A_{i_k} - 1$ are indeed elements of $M_n(P_n(I))^{GL_n(R)}$ and therefore $\mathcal{J}$ is well defined. By Proposition 5.10(1) the ideal $M_n(P_n(I))\mathcal{J} M_n(P_n(I)) \triangleleft M_n(P_n(I))$ is equal to $M_n(\mathcal{J}')$, where $\mathcal{J}' \triangleleft P_n(I)$ is the ideal generated by coefficients of matrices belonging to $\mathcal{J}$. Notice that $\mathcal{J}'$ is exactly the kernel of the epimorphism $\eta : P_n(I) \to Rep_n^R(G)$ introduced in Section 2. Therefore by Proposition 5.10 the homomorphism

$$M_n(\eta)^{GL_n(R)} : M_n(P_n(I))^{GL_n(R)} \to M_n(Rep_n^R(G))^{GL_n(R)}$$

considered in diagram (5.5) descends to an isomorphism

$$j : M_n(P_n(I))^{GL_n(R)} / \mathcal{J} \to M_n(Rep_n^R(G))^{GL_n(R)}.$$

Proposition 5.11. *$M_n(Rep_n^R(G))^{GL_n(R)}$ is generated by the elements $j_{G,n}(g_i)$ and $Tr\left(j_{G,n}(g_{i_1} g_{i_2} ... g_{i_k})\right)$, where $i, i_1, i_2, ..., i_k \in I$.*

Proof. From the paragraph preceding Proposition 5.11 it follows that $M_n(\eta)^{GL_n(R)}$ is an epimorphism. By Theorem 5.1, $M_n(P_n(I))^{GL_n(R)}$ is generated by the elements A_i and $Tr(A_{i_1} A_{i_2} ... A_{i_k})$, where $i, i_1, i_2, ..., i_k \in I$. The homomorphism $M_n(\eta)^{GL_n(R)}$ carries these elements to $j_{G,n}(g_i)$ and $Tr\left(j_{G,n}(g_{i_1} g_{i_2} ... g_{i_k})\right)$, respectively. □

This proposition and Theorem 3.6 imply that Θ is an epimorphism. We will show that it is also a monomorphism.

We have shown in Lemma 5.9 that $Det(A_i)-1 \in Ker\ \psi'$.[5] Moreover, by the definition of ψ', $A_{i_1}A_{i_2}...A_{i_k}-1 \in Ker\ \psi'$, for any $i_1, i_2, ..., i_k$ such that $g_{i_1}g_{i_2}...g_{i_k} = e$ in G. Therefore $\mathcal{J} \subset Ker\ \psi'$, and we can factor ψ' to

$$\psi'' : M_n(Rep_n^R(G))^{GL_n(R)} \to \mathbb{A}_n(X, x_0),$$

such that diagram (5.5) commutes and, by Corollary 5.8,

- $\psi''(j_{G,n}(g_i)) = E_{g_i}$,
- $\psi''(Tr(j_{G,n}(g_{i_1}g_{i_2}...g_{i_k})) = EL_{g_{i_1}g_{i_2}...g_{i_k}}$, for any $i_1, i_2, ..., i_k \in I$.

Recall that our assumptions about the presentation of G (stated in the paragraph following Theorem 5.3) say that the inverse of any generator of G is also a generator and that every element of G is a product of non-negative powers of generators. Thus, by Proposition 3.5(4), $\mathbb{A}_n(X, x_0)$ is generated by the elements E_{g_i} and $EL_{g_{i_1}g_{i_2}...g_{i_k}}$, for $i, i_1, i_2, ..., i_k \in I$. Since $\psi'' \circ \Theta$ is the identity on the generators of $\mathbb{A}_n(X, x_0)$, it also is the identity on $\mathbb{A}_n(X, x_0)$. Therefore Θ is a monomorphism.

In order to complete the proof of Theorem 3.7, we need to show that θ is also an isomorphism.

Fact 3.4 implies that we have an embedding $\imath_* : \mathbb{A}_n(X) \to \mathbb{A}_n(X, x_0)$, $\imath_*(L_g) = EL_g$, for $g \in G$. Therefore we can consider $\mathbb{A}_n(X)$ as a subring of $\mathbb{A}_n(X, x_0)$. Moreover, by Theorem 3.6, θ is just the restriction of

$$\Theta : \mathbb{A}_n(X, x_0) \to M_n(Rep_n^R(G))^{GL_n(R)}$$

to $\mathbb{A}_n(X)$. Therefore θ is a monomorphism.

In order to show that θ is an epimorphism we use once again an argument from invariant theory. By the naturality of the Reynolds operators $\nabla : M_n(Rep_n^R(G)) \to M_n(Rep_n^R(G))^{GL_n(R)}$ and $\nabla' : Rep_n^R(G) \to Rep_n^R(G)^{GL_n(R)}$, the following diagram commutes:

$$\begin{array}{ccc} M_n(Rep_n^R(G)) & \xrightarrow{Tr} & Rep_n^R(G) \\ \downarrow \nabla & & \downarrow \nabla' \\ M_n(Rep_n^R(G))^{GL_n(R)} & \xrightarrow{Tr} & Rep_n^R(G)^{GL_n(R)} \end{array}$$

Since $Tr : M_n(Rep_n^R(G)) \to Rep_n^R(G)$ and all Reynolds operators are epimorphic, $Tr : M_n(Rep_n^R(G))^{GL_n(R)} \to Rep_n^R(G)^{GL_n(R)}$ is also epimorphic. But now commutativity of (3.1) implies that θ is an epimorphism as well.

Therefore we have shown that θ is an isomorphism. This completes the proof of Theorem 3.7.

6. SL_n-CHARACTER VARIETIES

In this section we present one of the possible applications of Theorem 3.7 to a study of SL_n-character varieties.

Let X be a path-connected topological space whose fundamental group $G = \pi_1(X)$ is finitely generated. Let K be an algebraically closed field of characteristic 0. Recall that we noticed in Section 2 that the set of all $SL_n(K)$-characters of G, denoted by $X_n(G)$, is an algebraic set whose coordinate ring is $Rep_n^R(G)^{GL_n(K)}/\sqrt{0}$.

Let $\chi_g = Tr(j_{G,n}(g)) \in Rep_n^R(G)^{GL_n(K)}/\sqrt{0}$, for any $g \in G$. It is not difficult to check that χ_g, considered as an element of $K[X_n(G)]$, is a function which

[5] Recall that the map ψ' was defined in Corollary 5.8.

assigns to a character χ the value $\chi(g)$. By Proposition 3.5(3) and Theorem 3.7, $Rep_n^R(G)^{GL_n(K)}$ is generated by the elements $Tr(j_{G,n}(g))$. Therefore the functions χ_g generate $K[X_n(G)]$.

By an SL_n-trace identity for G we mean a polynomial function in variables $\chi_g, g \in G$, which is identically equal to 0 on $X_n(G)$. For example,

$$\chi_g \chi_h = \chi_{gh} + \chi_{gh^{-1}}$$

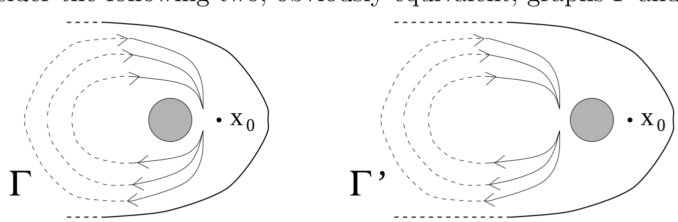
is the famous Fricke-Klein SL_2-trace identity, valid for any group G and any $g, h \in G$. From the above discussion it follows that the coordinate ring of $X_n(G)$ can be considered as the quotient of the ring of polynomials in formal variables $\chi_g, g \in G$, by the ideal of all SL_n-trace identities for G. Therefore Theorem 3.7 implies the following corollary.

Corollary 6.1. *There is an isomorphism $\Lambda : \mathbb{A}_n(X)/\sqrt{0} \to K[X_n(G)]$ such that $\Lambda(L_g) = \chi_g$. Under this isomorphism the identities on graphs in X induced by skein relations correspond to SL_n-trace identities for G.*

The above corollary is very useful in the study of trace identities, since it makes it possible to interpret them geometrically. Consider for example the following SL_3-trace identity, which holds for any $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in G$ and any $\chi \in X_3(G)$, where G is an arbitrary group:

$$\begin{aligned} &\chi(\gamma_1)\chi(\gamma_2)\chi(\gamma_3) - \chi(\gamma_1)\chi(\gamma_2\gamma_3) - \chi(\gamma_2)\chi(\gamma_1\gamma_3) - \chi(\gamma_3)\chi(\gamma_1\gamma_2) \\ &\quad + \chi(\gamma_1\gamma_2\gamma_3) + \chi(\gamma_1\gamma_3\gamma_2) - \chi(\gamma_1\gamma_0)\chi(\gamma_2\gamma_0)\chi(\gamma_3\gamma_0) \\ &\quad + \chi(\gamma_1\gamma_0)\chi(\gamma_2\gamma_0\gamma_3\gamma_0) + \chi(\gamma_2\gamma_0)\chi(\gamma_1\gamma_0\gamma_3\gamma_0) + \chi(\gamma_3\gamma_0)\chi(\gamma_1\gamma_0\gamma_2\gamma_0) \\ &\quad - \chi(\gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0) - \chi(\gamma_1\gamma_0\gamma_3\gamma_0\gamma_2\gamma_0) = 0. \end{aligned} \tag{6.1}$$

Our theory provides the following interpretation of this identity. Let $x_0 \in X$ and $G = \pi_1(X, x_0)$. Let γ_0 be a path in X representing a non-trivial element of $\pi_1(X, x_0)$. We assume that γ_0 goes along a "hole" in X presented in the picture below. Consider the following two, obviously equivalent, graphs Γ and Γ' in X:



The graph Γ' is obtained from Γ by pulling its vertices along the "hole" in X. The obvious resolution of vertices in Γ and Γ' gives an equation involving closed loops in X. This equation corresponds to the trace identity (6.1).

There is a large body of literature about SL_2-character varieties and their applications. However, very little is known about SL_n-character varieties for $n > 2$. The reason for this is that the SL_n-trace identities, like (6.1), are intractable by classical (algebraic) methods. Since our theory often gives a simple geometric interpretation to complicated trace identities, it can be applied to a more detailed study of character varieties. This idea was already used in [PS-2] and [PS-3] to study SL_n-character varieties for $n = 2$. A generalization of these results for $n > 2$, which is based on our skein method, will appear in future papers. In this work we test our method on the simplest non-trivial example – we study SL_3-character variety of the free group on two generators, $F_2 = \langle g_1, g_2 \rangle$. The basic problem is to determine the minimal dimension of the affine space in which $X_3(F_2)$ is embedded, or equivalently, the minimal number of generators of $K[X_3(F_2)]$. A result of Procesi (Theorem 3.4(a) [Pro-1]) implies that $K[X_3(F_2)]$ is generated by the elements $\chi_{g_{i_1} g_{i_2} \dots g_{i_j}}$, where $j \le 7$ and $i_1, i_2, ..., i_j \in \{1, 2\}$. A direct calculation shows that, after identifying words in g_1, g_2 which are related by cyclic permutations, we get a set of 57 generators of $K[X_3(F_2)]$. It is difficult to obtain any further reduction of this set in any simple algebraic manner. However, our geometric method allows us to reduce this problem to the study of 3-valent graphs in the twice-punctured disc. By playing with pictures of such graphs one can reduce the number of generators of $K[X_3(F_2)]$ to nine! These are

$$\chi_{g_1},\ \chi_{g_2},\ \chi_{g_1^2},\ \chi_{g_2^2},\ \chi_{g_1 g_2},\ \chi_{g_1^2 g_2},\ \chi_{g_1 g_2^2},\ \chi_{g_1^2 g_2^2},\ \chi_{g_1^2 g_2^2 g_1 g_2}.$$

Moreover, it is possible to show that this is the minimal number of generators, and $X_3(F_2) \subset K^9$ is a solution set of one irreducible polynomial.

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